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2007 J. Phys. A: Math. Theor. 40 8873

(http://iopscience.iop.org/1751-8121/40/30/017)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 8873–8901 doi:10.1088/1751-8113/40/30/017

# On character generators for simple Lie algebras

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Received 6 February 2007, in final form 10 May 2007
Published 12 July 2007
Online at stacks.iop.org/JPhysA/40/8873

**Abstract**
We study character generating functions (character generators) of simple Lie algebras. The expression due to Patera and Sharp, derived from the Weyl character formula, is first reviewed. A new general formula is then found. It makes clear the distinct roles of 'outside' and 'inside' elements of the integrity basis, and helps determine their quadratic incompatibilities. We review, analyse and extend the results obtained by Gaskell using the Demazure character formulae. We find that the fundamental generalized-poset graphs underlying the character generators can be deduced from such calculations. These graphs, introduced by Baclawski and Towber, can be simplified for the purposes of constructing the character generator. The generating functions can be written easily using the simplified graphs, and associated Demazure expressions. The rank-two algebras are treated in detail, but we believe our results are indicative of those for general simple Lie algebras.

PACS number: 02.20.Sv
Mathematics Subject Classification: 17B10, 17B81

## 1. Introduction

Characters are important tools in the representation theory of Lie groups and algebras, and so are relevant to many physical applications. Generating function techniques are powerful and general, and have been usefully applied in many areas of mathematics and mathematical physics.

Combining characters and generating functions leads to the study of character generators [21], the generating functions of characters. We will report results on the character generators for the irreducible, integrable, highest-weight representations of finite-dimensional, simple Lie algebras.

Much work has been done on these character generators, and on other generating functions relevant to Lie algebras (see [22] for a brief summary). Our emphasis will be on general results, valid for all simple Lie algebras and expressed in terms of structures common to them[1]. Relevant highlights of past research include the papers by Patera and Sharp [21], Stanley [23], King [15], Baclawski [1], King and El-Sharkaway [16, 17], and Baclawski and Towber [2].

Patera and Sharp [21] introduced the character generator, and used the Weyl character formula to write a general formula for it. While it is general, the Patera–Sharp formula has the same drawbacks as its summand, the Weyl character formula. It involves positive and negative terms that cancel, and a sum over the Weyl group. The Weyl group sum is enormous for all but the lowest rank algebras. In the first of two parts of this paper, starting from the Patera–Sharp formula, we will derive a general formula for the character generator that does not involve that sum.

The first drawback still applies to the new formula, however. To write non-negative formulae, a more microscopic point of view helps. The character generator is a generating function for characters, but characters are themselves generating functions for weight multiplicities. Combinatorial methods for the calculation of these multiplicities are well known. In particular, methods involving Young tableaux and variants are very efficient. Stanley [23] was the first to apply them to the character generator. In particular, he proved a formula for the $A_r \cong su(r+1)$ character generator involving standard shifted Young tableaux. Soon afterwards, King [15] adapted Stanley's work to the case $C_r \cong \mathrm{sp}(2r)$.

Most importantly for us, these standard shifted Young tableaux encode a partially ordered set, or poset. The poset structure of the character generator was made explicit by Baclawski [1]. He proved combinatorial formulae for the character generators of $A_r$, $C_r$ (and $U(N)$) in terms of poset objects related to the relevant tableaux.

King and El-Sharkaway generalized these results to all the classical algebras in [16, 17], using generalized standard Young tableaux. This work is very, but not completely, general: besides the classical simple Lie algebras $A_r$, $B_r$, $C_r$, $D_r$, there are also the exceptional ones $E_6$, $E_7$, $E_8$, $F_4$, $G_2$. The notion of standard Young tableau was not general enough to include the exceptional Lie algebras[2].

Baclawski and Towber [2] studied the simplest exceptional algebra, $G_2$. They were able to construct the character generator, and found that a generalization of a poset was relevant. This generalized-poset structure is important, but it was revealed by a construction specific to $G_2$, related to the octonions. A truly general construction was therefore not found.

Meanwhile, however, new general methods were applied to the problem. The Demazure character formulae [6] are valid for all simple Lie algebras (and others), and lead directly to non-negative expressions for the characters. Gaskell [9] wrote a general formula for the character generator as a Demazure operator acting on the highest-weight generating function (see (83)), and calculated several examples[3].

In the second part of this paper, we will push Gaskell's methods, and apply Demazure character formulae to the character generator. Most importantly, we will also search for the underlying generalized-poset structure of the character generator. By combining the Gaskell–Demazure techniques with the generalized-poset structure, we make progress. While our calculations focus on the simple Lie algebras of rank two, we believe that our results indicate those for all simple Lie algebras. We are able to conjecture a general non-negative formula for the character generator of an arbitrary simple Lie algebra.

[1] Such relations are sometimes called universal.

[2] More recently, however, Littelmann has provided such generalizations, in the form of minimal defining chains of elements of the Weyl group [18], and also in terms of so-called Lakshmibai–Seshadri paths and root operators [19].

[3] Actually, Gaskell was unaware of Demazure's work. Remarkably, he re-discovered some of the Demazure formulae independently, in order to apply them to character generators.

The layout of the paper is as follows. In the following section, we first review the derivation of the Patera–Sharp formula from the Weyl character formula, and then derive a new general formula from the Patera–Sharp one. It makes clear the distinct roles of the inside and outside generators of the integrity basis for the terms of the character generator. As we show in section 3, it can also serve as a guide to the incompatible products (consequences of syzygies) of the elements of the integrity basis. In section 4, we review the Demazure character formulae, and apply them to the construction of character generators, following Gaskell. Simple, non-negative expressions are found for the character generators of all the rank-two simple Lie algebras. In section 5, the poset and generalized-poset structures of character generators are reviewed, and then applied to our rank-two results. We simplify the generalized-poset graphs introduced by Baclawski and Towber, and introduce edge labels for the new graphs (dubbed character-generator graphs). These edge labels are expressed in terms of Demazure operators, and so can be simply determined. Consequently, we are able to write a formula that is valid for all the rank-two algebras, and others, in terms of a so-called fundamental-orbit poset and Demazure quantities. We conjecture that this formula is universal, i.e. applicable to all simple Lie algebras. Section 6 is our conclusion.

## 2. Character generators from the Weyl character formula

Let $\mathrm{X}(L, a)$ denote the generator (generating function) for the characters of a fixed simple Lie algebra $X_r$, of rank $r$. It is defined by [21]

$$\mathrm{X}(L, a) := \sum_{\lambda\in P_{\geqslant}} L^{\lambda}\, \mathrm{ch}_{\lambda}(a), \tag{1}$$

where the character of the integrable, irreducible representation $R(\lambda)$ of highest weight $\lambda$ is

$$\mathrm{ch}_{\lambda}(a) = \sum_{\sigma\in P} \mathrm{mult}_{\lambda}(\sigma) a^{\sigma}. \tag{2}$$

Two sets of indeterminate variables are used. We write

$$L^{\lambda} = L^{\sum_i \lambda_i \Lambda^i} := L_1^{\lambda_1} \cdots L_r^{\lambda_r} \tag{3}$$

to keep track of the highest weights of representations, and $a^{\mu} := a_1^{\mu_1} \cdots a_r^{\mu_r}$ to record the weights with nonvanishing multiplicities in those representations. In (1), $\mathrm{mult}_{\lambda}(\sigma)$ is the multiplicity of weight $\sigma$ in $R(\lambda)$.

The fundamental weights are the $\Lambda^j$, and the set thereof will be denoted by $F$. The set of integral weights of $X_r$ is

$$P := \left\{ \sum_{i=1}^{r} \lambda_i \Lambda^i | \lambda_i \in \mathbb{Z} \right\}, \tag{4}$$

i.e., the set of weights with integer Dynkin labels $\lambda_i$. $P_{\geqslant} \subset P$ will be the set of dominant weights

$$P_{\geqslant} := \left\{ \sum_{i=1}^{r} \lambda_i \Lambda^i | \lambda_i \in \mathbb{Z}_{\geqslant} \right\}, \tag{5}$$

with non-negative integer (semi-natural) Dynkin labels. Similarly, $P_{>} \subset P$ will denote the set of weights

$$P_{>} := \left\{ \sum_{i=1}^{r} \lambda_i \Lambda^i | \lambda_i \in \mathbb{Z}_{>} \right\}, \tag{6}$$

with positive integer (natural) Dynkin labels. We will also sometimes use the notation

$$\lambda = \sum_{i=1}^{r} \lambda_i \Lambda^i =: (\lambda_1, \lambda_2, \ldots, \lambda_r). \tag{7}$$

The set of weights of representation $R(\lambda)$ will be indicated by

$$P_\lambda := \{\mu \in P | \mathrm{mult}_\lambda(\mu) \geqslant 1\}. \tag{8}$$

The Weyl formula for the character of $R(\lambda)$ is

$$\begin{aligned} \mathrm{ch}_\lambda(a) &= \sum_{w \in W} a^{w\lambda} \prod_{\alpha \in \Delta_+} (1 - a^{-w\alpha})^{-1} \\ &= \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \sum_{w \in W} (\det w) a^{w \cdot \lambda}. \end{aligned} \tag{9}$$

$W$ is the Weyl group of $X_r$, $\Delta_+$ the set of its positive roots, and $w \cdot \lambda = w(\lambda + \rho) - \rho$ is the shifted action of the Weyl group element $w \in W$. The Weyl vector is denoted as

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \Lambda^1 + \Lambda^2 + \cdots + \Lambda^r = \sum_{\Lambda \in F} \Lambda. \tag{10}$$

The key observation of [21] is that if the Weyl character formula (9) is used, the sum over $P_\geqslant$ in (1) can be done, yielding the Patera–Sharp formula

$$\mathrm{X}(L, a) = \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \sum_{w \in W} a^{w\rho - \rho} (\det w) \prod_{\Lambda \in F} (1 - L^\Lambda a^{w\Lambda})^{-1}. \tag{11}$$

This is already a nice, general result. However, the sum over the Weyl group is daunting for any but the smallest Lie algebras. Furthermore, division by the Weyl denominator $\prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1}$ makes direct computation quite difficult.

However, if we factor out a common denominator, call it Z, things improve somewhat. As is usual, let $W\lambda$ indicate the set of weights in the Weyl orbit of $\lambda$. Then we can write

$$\mathrm{X}(L, a) = \prod_{\Lambda \in F} \prod_{\varphi \in W\Lambda} (1 - L^\Lambda a^\varphi)^{-1} \mathrm{Y} =: \mathrm{Z}^{-1} \mathrm{Y}, \tag{12}$$

with

$$\mathrm{Y} = \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \sum_{w \in W} a^{w\rho - \rho} (\det w) \prod_{\Lambda \in F} \prod_{\sigma \in W\Lambda \backslash \{w\Lambda\}} (1 - L^\Lambda a^\sigma). \tag{13}$$

It is well known that the characters may be written as integer polynomials of the fundamental characters. We therefore expect that the integrity basis $I_\mathrm{X}$ will be

$$I_\mathrm{X} = \{L^\Lambda a^\varphi \mid \Lambda \in F, \varphi \in P_\Lambda\}. \tag{14}$$

That is, we expect that the character generator X can be written as a rational function of the elements of $I_\mathrm{X}$. For the integrity-basis element $L^\Lambda a^\varphi$, $\Lambda$ and $\varphi$ will be known as its shape (or its highest weight) and its weight, respectively.

Clearly, it is the numerator Y that encodes the truly nontrivial information carried by a character generator X. The denominator Z tells us only that the 'outside weights' of the fundamental representations determine a subset

$$I_\mathrm{out} = \{L^\Lambda a^\varphi \mid \Lambda \in F, \varphi \in W\Lambda\} \tag{15}$$

of the integrity basis $I_\mathrm{X}$ for the terms of X. The elements of $I_\mathrm{out}$ and $I_\mathrm{in} := I_\mathrm{X} \backslash I_\mathrm{out}$ will be called outside and inside generators, respectively.

Another helpful observation is that the terms in the sum of (13) are simply related to each other. Let $\hat{w}$ denote an 'operator' with action

$$\hat{w}(L^{\nu}a^{\mu}) := L^{\nu}a^{w\mu}, \tag{16}$$

for any weights $\nu$, $\mu$. Then we can write

$$\mathrm{Y} = \prod_{\alpha\in\Delta_+}(1-a^{-\alpha})^{-1}\sum_{w\in W} a^{w\rho-\rho}(\det w)\hat{w}(\mathcal{Y}), \tag{17}$$

where we have defined

$$\mathcal{Y} := \prod_{\Lambda\in F}\prod_{\sigma\in\overline{W\Lambda}}(1-L^{\Lambda}a^{\sigma}) \tag{18}$$

and the shorthand

$$\overline{W\Lambda} := W\Lambda\backslash\{\Lambda\}. \tag{19}$$

Now, comparing with the Weyl formula (9), we see that

$$\begin{aligned}\hat{\mathrm{ch}} &:= \prod_{\alpha\in\Delta_+}(1-a^{-\alpha})^{-1}\sum_{w\in W} a^{w\rho-\rho}(\det w)\hat{w}\\ &= \sum_{w\in W}\hat{w}\prod_{\alpha\in\Delta_+}(1-a^{-\alpha})^{-1}\end{aligned} \tag{20}$$

acts as follows:

$$\hat{\mathrm{ch}}(a^{\lambda}) = \mathrm{ch}_{\lambda}(a). \tag{21}$$

Therefore, we get

$$\mathrm{Y} = \widehat{\mathrm{ch}}(\mathcal{Y}) = \hat{\mathrm{ch}}\left(\prod_{\Lambda\in F}\prod_{\sigma\in\overline{W\Lambda}}(1-L^{\Lambda}a^{\sigma})\right). \tag{22}$$

This formula shows that we can decompose Y into characters,

$$\mathrm{Y} = \sum_{\mu\in P_{\geqslant}}\mathrm{y}_{\mu}(L)\,\mathrm{ch}_{\mu}. \tag{23}$$

The coefficients $\mathrm{y}_{\mu}(L)$ will be polynomials in the $L_j = L^{\Lambda^j}$, with integer coefficients. To evaluate Y in this form, we use the shifted-Weyl (anti-)symmetry of the characters:

$$\mathrm{ch}_{\lambda} = (\det w)\,\mathrm{ch}_{w\cdot\lambda}. \tag{24}$$

If we define a partition function $K_{\mu}(L)$ as follows[4]:

$$\mathcal{Y}(L,a) = \prod_{\Lambda\in F}\prod_{\sigma\in\overline{W\Lambda}}(1-L^{\Lambda}a^{\sigma}) =: \sum_{\tau\in P}K_{\tau}(L)a^{\tau}, \tag{25}$$

then the desired coefficients can be computed using

$$\mathrm{y}_{\mu}(L) = \sum_{w\in W}(\det w)K_{w\cdot\mu}(L). \tag{26}$$

This equation says that the $\mathrm{y}_{\mu}$ can be calculated by first expanding $\mathcal{Y}$, using its definition (see equation (18)). Each term obtained with $a$-dependence $a^{\varphi}$ can be Weyl-transformed using the shifted action, so that the result $a^{\nu}$ has $\nu+\rho$ either in $P_{>}$, or on its boundary (i.e. having at least one vanishing Dynkin label). In the latter case, the term should be dropped. In the former case, it contributes with an extra factor of $\det w$, where $w$ is the Weyl group element

[4] Here we imitate the definition of the Kostant partition function. See section 25.2 of [8], for example.

used. All terms $a^{\nu}$ so collected, with $\nu \in P_{\geqslant}$, signal a contribution of $\mathrm{ch}_\nu$ to Y. We hope that the examples worked through in the following section will make the procedure clear.

Formally, then, the answer is

$$\mathrm{Y}(L, a) = \sum_{\nu \in P_{\geqslant}} \mathrm{ch}_\nu(a) \sum_{w \in W} (\det w) K_{w\cdot\nu}(L), \tag{27}$$

so that the character generator is

$$\mathrm{X}(L, a) = \left\{ \prod_{\substack{\Lambda \in F \\ \varphi \in W\Lambda}} (1 - L^{\Lambda} a^{\varphi}) \right\}^{-1} \sum_{\nu \in P_{\geqslant}} \mathrm{ch}_\nu(a) \sum_{w \in W} (\det w) K_{w\cdot\nu}(L). \tag{28}$$

### 2.1. Examples

*2.1.1.* $A_1$. For $X_r = A_1$, there is only one fundamental weight, $\Lambda^1$, and we have

$$\mathrm{Z} = (1 - L^{\Lambda^1} a^{\Lambda^1})(1 - L^{\Lambda^1} a^{-\Lambda^1}). \tag{29}$$

Since $\mathcal{Y} = (1 - L^{\Lambda^1} a^{-\Lambda^1})$, we have

$$\mathrm{Y} = 1 - L^{\Lambda^1}\, \mathrm{ch}_{-\Lambda^1}, \tag{30}$$

by (22). But $\mathrm{ch}_{-\Lambda^1} = -\mathrm{ch}_{-\Lambda^1} = 0$, by (24), so that $\mathrm{Y} = 1$. Finally, we have the well-known result

$$\mathrm{X}(L, a) = \left[\left(1 - L^{\Lambda^1} a^{\Lambda^1}\right)\left(1 - L^{\Lambda^1} a^{-\Lambda^1}\right)\right]^{-1}. \tag{31}$$

*2.1.2.* $A_2$.

$$\begin{aligned} \mathrm{Z} = {} & \left(1 - L^{\Lambda^1} a^{\Lambda^1}\right)\left(1 - L^{\Lambda^1} a^{-\Lambda^1+\Lambda^2}\right)\left(1 - L^{\Lambda^1} a^{-\Lambda^2}\right) \\ & \times \left(1 - L^{\Lambda^2} a^{\Lambda^2}\right)\left(1 - L^{\Lambda^2} a^{\Lambda^1-\Lambda^2}\right)\left(1 - L^{\Lambda^2} a^{-\Lambda^1}\right), \end{aligned} \tag{32}$$

and

$$\mathcal{Y} = \left(1 - L^{\Lambda^1} a^{-\Lambda^1+\Lambda^2}\right)\left(1 - L^{\Lambda^1} a^{-\Lambda^2}\right)\left(1 - L^{\Lambda^2} a^{\Lambda^1-\Lambda^2}\right)\left(1 - L^{\Lambda^2} a^{-\Lambda^1}\right). \tag{33}$$

From (22), expanding $\mathcal{Y}$ and applying $\widehat{\mathrm{ch}}$ gives

$$\begin{aligned} \mathrm{Y} = {} & 1 - L^{\Lambda^1}(\mathrm{ch}_{-\Lambda^1+\Lambda^2} + \mathrm{ch}_{-\Lambda^2}) - L^{\Lambda^2}(\mathrm{ch}_{\Lambda^1-\Lambda^2} + \mathrm{ch}_{-\Lambda^1}) + L^{2\Lambda^1}\,\mathrm{ch}_{-\Lambda^2} + L^{2\Lambda^2}\,\mathrm{ch}_{-\Lambda^2} \\ & + L^{\Lambda^1+\Lambda^2}(1 + \mathrm{ch}_{-2\Lambda^1+\Lambda^2} + \mathrm{ch}_{\Lambda^1-2\Lambda^2} + \mathrm{ch}_{-\Lambda^1-\Lambda^2}) \\ & - L^{2\Lambda^1+\Lambda^2}(\mathrm{ch}_{-\Lambda^2} + \mathrm{ch}_{-2\Lambda^1}) - L^{\Lambda^1+2\Lambda^2}(\mathrm{ch}_{-\Lambda^1} + \mathrm{ch}_{-2\Lambda^2}) \\ & + L^{2\Lambda^1+2\Lambda^2}\,\mathrm{ch}_{-\Lambda^1-\Lambda^2}. \end{aligned} \tag{34}$$

Any term $\mathrm{ch}_\mu$ with a Dynkin label $\mu_i = -1$ vanishes, since if $r_i$ denotes the primitive reflection related to the simple root $\alpha_i$, then $r_i \cdot \mu = \mu$. Equation (24) then tells us that $\mathrm{ch}_\mu = -\mathrm{ch}_\mu = 0$. The expression immediately simplifies to

$$\mathrm{Y} = 1 + L^{\Lambda^1+\Lambda^2}(1 + \mathrm{ch}_{-2\Lambda^1+\Lambda^2} + \mathrm{ch}_{\Lambda^1-2\Lambda^2}) - L^{2\Lambda^1+\Lambda^2}\,\mathrm{ch}_{-2\Lambda^1} - L^{\Lambda^1+2\Lambda^2}\,\mathrm{ch}_{-2\Lambda^2}. \tag{35}$$

But $r_1 \cdot (-2\Lambda^1) = -\Lambda^2$ and $r_2 \cdot (-2\Lambda^2) = -\Lambda^1$, so the last two terms vanish. Also, $r_1 \cdot (-2\Lambda^1 + \Lambda^2) = r_2 \cdot (\Lambda^1 - 2\Lambda^2) = 0$, so that we obtain

$$\mathrm{Y} = 1 - L^{\Lambda^1+\Lambda^2}. \tag{36}$$

Finally, we can write

$$\mathrm{X} = \mathrm{Z}^{-1}\left[1 - L^{\Lambda^1+\Lambda^2}\right], \tag{37}$$

with Z given by (32), in agreement with the known result [21].

*2.1.3. $B_2$.* The simple roots are $\alpha_1 = 2\Lambda^1 - 2\Lambda^2$ and $\alpha_2 = -\Lambda^1 + 2\Lambda^2$. The Weyl orbits of the fundamental weights,

$$\begin{aligned} W\Lambda^1 &= \{\pm\Lambda^1, \pm(-\Lambda^1 + 2\Lambda^2)\} \\ W\Lambda^2 &= \{\pm\Lambda^2, \pm(\Lambda^1 - \Lambda^2)\}, \end{aligned} \tag{38}$$

determine both Z and $\mathcal{Y}$ immediately. By (22), we find

$$\mathrm{Y} = 1 + L^{\Lambda^1} - L^{\Lambda^1+\Lambda^2}\,\mathrm{ch}_{\Lambda^2} + L^{\Lambda^1+2\Lambda^2} + L^{2\Lambda^1+2\Lambda^2}. \tag{39}$$

Using

$$\mathrm{ch}_{\Lambda^2} = a^{\Lambda^2} + a^{-\Lambda^1+\Lambda^2} + a^{-\Lambda^2} + a^{\Lambda^1-\Lambda^2}, \tag{40}$$

we have checked that this answer agrees with the known result [21].

*2.1.4. $G_2$.* The simple roots are $\alpha_1 = 2\Lambda^1 - 3\Lambda^2$ and $\alpha_2 = -\Lambda^1 + 2\Lambda^2$. The Weyl orbits of the fundamental weights are

$$\begin{aligned} W\Lambda^1 &= \{\pm\Lambda^1, \pm(\Lambda^1 - 3\Lambda^2), \pm(2\Lambda^1 - 3\Lambda^2)\}, \\ W\Lambda^2 &= \{\pm\Lambda^2, \pm(\Lambda^1 - \Lambda^2), \pm(\Lambda^1 - 2\Lambda^2)\}. \end{aligned} \tag{41}$$

By (22), expanding $\mathcal{Y}$ and applying $\widehat{\mathrm{ch}}$ gives

$$\begin{aligned} Y = 1 &+ L^{\Lambda^1} + L^{\Lambda^2} + L^{3\Lambda^1+3\Lambda^2} + L^{\Lambda^1+4\Lambda^2} + L^{3\Lambda^1} + L^{4\Lambda^1+3\Lambda^2} + L^{\Lambda^1+\Lambda^2} + L^{\Lambda^1+3\Lambda^2} \\ &+ L^{\Lambda^1+2\Lambda^2} + L^{3\Lambda^1+2\Lambda^2} + L^{4\Lambda^1+4\Lambda^2} + L^{2\Lambda^1+4\Lambda^2} + L^{3\Lambda^1+4\Lambda^2} + L^{3\Lambda^1+\Lambda^2} + L^{2\Lambda^1} \\ &+ \left(L^{3\Lambda^1+4\Lambda^2} + L^{2\Lambda^1+4\Lambda^2} + L^{\Lambda^1} + L^{2\Lambda^1} + 2L^{2\Lambda^1+2\Lambda^2} - L^{3\Lambda^1+\Lambda^2} - L^{\Lambda^1+3\Lambda^2}\right)\mathrm{ch}_{\Lambda^2} \\ &+ \left(L^{3\Lambda^1+2\Lambda^2} + L^{\Lambda^1+2\Lambda^2} - L^{2\Lambda^1+\Lambda^2} - L^{2\Lambda^1+3\Lambda^2}\right)\mathrm{ch}_{\Lambda^1} \\ &+ L^{2\Lambda^1+2\Lambda^2}\,\mathrm{ch}_{\Lambda^1+\Lambda^2} - \left(L^{\Lambda^1+\Lambda^2} + L^{2\Lambda^1+\Lambda^2} + L^{3\Lambda^1+3\Lambda^2} + L^{2\Lambda^1+3\Lambda^2}\right)\mathrm{ch}_{2\Lambda^2} \end{aligned} \tag{42}$$

after applying (24). When the required characters are substituted, this expression agrees with the known result [12].

## 3. Integrity basis and incompatibilities

The characters can be generated by an integrity basis subject to certain relations. The basis is given in (14). For a fixed simple Lie algebra, the character of any irreducible representation can be written as a non-negative integer polynomial in these basis elements.

Important relations can be expressed as incompatibilities, quadratic products of basis elements that do not appear in any of the monomials just mentioned. Here we show how the new formula (28) can be used as a guide to the incompatibilities. A different method was described in [13].

Since the outside and inside generators ($I_{\mathrm{in}} := I_{\mathrm{X}}\backslash I_{\mathrm{out}}$) play different roles in generating characters, it will be useful to split the fundamental characters into inside and outside parts, by writing

$$\mathrm{ch}_{\Lambda}(a) =: \mathcal{O}_{\Lambda}(a) + \mathcal{I}_{\Lambda}(a). \tag{43}$$

Here we denote the orbit sum by

$$\mathcal{O}_{\lambda}(a) = \sum_{\sigma\in W\lambda} a^{\sigma}. \tag{44}$$

The numerator Y contains the required information. Using (22), we will expand Y up to terms quadratic in the $L_i$:

$$\mathrm{Y} = \mathrm{Y}^{(0)} + \mathrm{Y}^{(1)} + \mathrm{Y}^{(2)} + \cdots = \widehat{\mathrm{ch}}(\mathcal{Y}^{(0)} + \mathcal{Y}^{(1)} + \mathcal{Y}^{(2)} + \cdots). \tag{45}$$

Clearly,

$$\mathrm{Y}^{(0)} = \mathcal{Y}^{(0)} = 1. \tag{46}$$

The linear term is

$$\mathrm{Y}^{(1)} = -\widehat{\mathrm{ch}}\left(\sum_{\Lambda\in F}\sum_{\phi\in\overline{W\Lambda}} L^\Lambda a^\phi\right) = \widehat{\mathrm{ch}}\left(\sum_{\Lambda\in F} L^\Lambda(a^\Lambda - \mathcal{O}_\Lambda(a))\right); \tag{47}$$

see (44). We now use the identity

$$\widehat{\mathrm{ch}}(a^\mu \mathcal{O}_\lambda(a)) = \mathrm{ch}_\mu(a)\mathcal{O}_\lambda(a), \tag{48}$$

which can be proved easily from the second line of (20). Here we only need to consider $\mu = 0$, and we find

$$\begin{aligned}\mathrm{Y}^{(1)} &= \sum_{\Lambda\in F} L^\Lambda(\mathrm{ch}_\Lambda(a) - \mathcal{O}_\Lambda(a)) \\ &=: \sum_{\Lambda\in F} L^\Lambda \mathcal{I}_\Lambda.\end{aligned} \tag{49}$$

The result shows explicitly that the inside generators all appear linearly in X:

$$\mathrm{Y}^{(1)} = \sum_{\iota\in I_{\mathrm{in}}} \iota, \tag{50}$$

for all simple Lie algebras.

The quadratic term can be expressed as

$$\mathcal{Y}^{(2)} = \sum_{\Lambda\in F} L^{2\Lambda} \sum_{\substack{\phi,\phi'\in\overline{W\Lambda}\\ \phi'\neq\phi}} a^{\phi+\phi'} + \sum_{\substack{\Lambda,\Lambda'\in F\\ \Lambda\neq\Lambda'}} L^{\Lambda+\Lambda'} \sum_{\substack{\phi\in\overline{W\Lambda}\\ \phi'\in\overline{W\Lambda'}}} a^{\phi+\phi'}. \tag{51}$$

This leads to the expression

$$\mathrm{Y}^{(2)} = \sum_{\substack{\Lambda,\Lambda'\in F\\ \Lambda'\neq\Lambda}} (\mathcal{I}_\Lambda\mathcal{I}_{\Lambda'} - S_{\Lambda,\Lambda'})L^{\Lambda+\Lambda'} + \sum_{\Lambda\in F}\left(\mathrm{ch}_{2\Lambda} - S_{\Lambda,\Lambda} + \mathcal{I}_\Lambda^2 - \mathcal{O}_{2\Lambda}\right)L^{2\Lambda}. \tag{52}$$

Here we have defined

$$S_{\Lambda,\Lambda'} := \mathrm{ch}_\Lambda\,\mathrm{ch}_{\Lambda'} - \mathrm{ch}_{\Lambda+\Lambda'}. \tag{53}$$

Similar expressions for terms $\mathrm{Y}^{(n)}$ with $n > 2$ are complicated. We will focus on the quadratic term $\mathrm{Y}^{(2)}$ below, and treat each of the rank-two simple Lie algebras in turn.

### *3.1. Examples*

*3.1.1. $A_2$.* From subsection 2.1.2,

$$\mathrm{Y}^{(2)} = -L_1L_2. \tag{54}$$

This result agrees with that calculated using (51).

For any algebra $A_r$, $\mathcal{I}_\Lambda = 0$ for all $\Lambda \in F$. Using

$$\begin{aligned}&\mathrm{ch}_{\Lambda^1}\,\mathrm{ch}_{\Lambda^2} = \mathrm{ch}_{\Lambda^1+\Lambda^2} + 1, \\ &(\mathrm{ch}_{\Lambda^1})^2 = \mathrm{ch}_{2\Lambda^1} + \mathrm{ch}_{\Lambda^2}, \qquad (\mathrm{ch}_{\Lambda^2})^2 = \mathrm{ch}_{2\Lambda^2} + \mathrm{ch}_{\Lambda^1}, \\ &\mathrm{ch}_{2\Lambda^1} - \mathcal{O}_{2\Lambda^1} = \mathrm{ch}_{\Lambda^2}, \qquad \mathrm{ch}_{2\Lambda^2} - \mathcal{O}_{2\Lambda^2} = \mathrm{ch}_{\Lambda^1},\end{aligned} \tag{55}$$

(52) gives the same result.

The interpretation of the result (54) is simple. There is one incompatible quadratic product, but it is not uniquely determined. Any of the three following possibilities works:

$$(L_1a_1)\big(L_2a_1^{-1}\big), \qquad \big(L_1a_1^{-1}a_2\big)\big(L_2a_1a_2^{-1}\big), \qquad \big(L_1a_2^{-1}\big)(L_2a_2). \tag{56}$$

We will see in subsection 4.1.1 that these choices lead to three different, but equivalent, expressions for X.

*3.1.2. $B_2$.* To use (52), we need

$$\begin{aligned} &\mathcal{I}_{\Lambda^1} = 1, \qquad \mathcal{I}_{\Lambda^2} = 0, \\ &S_{\Lambda^1,\Lambda^1} = 1 + \mathrm{ch}_{2\Lambda^2}, \qquad S_{\Lambda^1,\Lambda^2} = \mathrm{ch}_{\Lambda^2}, \qquad S_{\Lambda^2,\Lambda^2} = 1 + \mathrm{ch}_{\Lambda^1}, \\ &\mathrm{ch}_{2\Lambda^1} - \mathcal{O}_{2\Lambda^1} = \mathrm{ch}_{2\Lambda^2}, \qquad \mathrm{ch}_{2\Lambda^2} - \mathcal{O}_{2\Lambda^2} = 1 + \mathrm{ch}_{\Lambda^1}, \end{aligned} \tag{57}$$

to find

$$\mathrm{Y}^{(2)} = -L_1L_2\,\mathrm{ch}_{\Lambda^2}. \tag{58}$$

This is in agreement with the result of subsection 2.1.3.

The negative terms in (58) reveal incompatibilities between generators. One choice of incompatible products is

$$(L_1a_1)\big(L_2a_1^{-1}a_2\big), \qquad (L_1a_1)\big(L_2a_2^{-1}\big), \qquad \big(L_1a_1^{-1}a_2^2\big)\big(L_2a_2^{-1}\big), \qquad (L_1)\big(L_2a_2^{-1}\big). \tag{59}$$

The sum of these four terms equals $L_1L_2\,\mathrm{ch}_{\Lambda^2}$, therefore agreeing with (58).

In subsection 4.1.2, we will relate this choice of incompatible products to a non-negative expression for X, and an underlying graph.

*3.1.3. $G_2$.* From subsection 2.1.4,

$$\mathrm{Y}^{(2)} = L_1^2(1 + \mathrm{ch}_{\Lambda^2}) + L_1L_2(1 - \mathrm{ch}_{2\Lambda^2}). \tag{60}$$

This result can be seen to agree with that calculated using (52).

We will verify in subsection 4.1.3 that the expression (60) encodes the incompatible products for X. More precisely, we will show that it can be written as a sum of terms

$$\mathrm{Y}^{(2)} = -\mathrm{Y}^{(2)}_{\mathrm{out,out}} + \mathrm{Y}^{(2)}_{\mathrm{in,in}} - \mathrm{Y}^{(2)}_{\mathrm{in,out}}. \tag{61}$$

The negative terms are incompatible products, either with two outer generators as factors, or one inner and one outer. Since the factor $Z^{-1}$ of X does not involve the inner generators, the allowed products quadratic in the inner generators appear in $\mathrm{Y}^{(2)}$; that explains the positive term.

## 4. Gaskell character generators from Demazure character formulae

In this section, we follow Gaskell [11] and apply the Demazure character formulae to the calculation of character generators. We will be able to interpret our results in terms of certain graphs, as discussed in section 5.

Let us first review the Demazure character formula(e), and set our notation. Demazure [6] introduced the operators $\widehat{D}_i$, $i = 1, \ldots, r$, associated with the simple roots of the Lie algebra $X_r$, or the corresponding primitive reflections $r_i$. They are defined by the action

$$\widehat{D}_i(a^\phi) = \begin{cases} a^\phi + a^{\phi-\alpha_i} + \cdots + a^{\phi-\phi_i\alpha_i}, & \phi_i \geqslant 0; \\ 0, & \phi_i = -1; \\ -a^{\phi+\alpha_i} - a^{\phi+2\alpha_i} - \cdots - a^{\phi+(|\phi_i|-1)\alpha_i}, & \phi_i \leqslant -2. \end{cases} \tag{62}$$

The number of terms in these expansions is $|\phi_i + 1|$. Alternatively, one can write

$$\widehat{D}_i = (1 - a^{-\alpha_i})^{-1}(1 - a^{-\alpha_i}\hat{r}_i). \tag{63}$$

A unique Demazure operator can be defined for every element of the Weyl group $W$. Suppose $w \in W$ has a reduced decomposition $w = s_\ell \cdots s_2 s_1$. Here each $s_j = r_{j'}$ is a primitive reflection. Since the decomposition is reduced, $\ell$ is the minimum possible length, the length $\ell(w)$ of $w$. Then we can define

$$\widehat{D}_w := \widehat{D}_{1'}\widehat{D}_{2'} \cdots \widehat{D}_{\ell'}. \tag{64}$$

Reduced decompositions are not unique, however. For example, the longest element $w_L$ of the $su(3)$ Weyl group $W \cong S_3$ has two such decompositions:

$$w_L = r_1 r_2 r_1 = r_2 r_1 r_2. \tag{65}$$

But the braid relation that equates them is also satisfied by the Demazure operators:

$$\widehat{D}_{w_L} = \widehat{D}_1\widehat{D}_2\widehat{D}_1 = \widehat{D}_2\widehat{D}_1\widehat{D}_2, \tag{66}$$

so that $\widehat{D}_{w_L}$ can be constructed using either of its reduced decompositions. Such braid relations are obeyed for any simple Lie algebra, and the operators $\widehat{D}_w$ are uniquely defined for any $w \in W$. The basic operators are the $\widehat{D}_i := \widehat{D}_{r_i}$.

Note, however, that although the braid relations of the Weyl group are obeyed by the Demazure operators, we have $r_i^2 = 1$, but $\widehat{D}_i^2 \neq 1$. Instead

$$(\widehat{D}_i)^2 = \widehat{D}_i, \tag{67}$$

so that the $\widehat{D}_i$ are projection operators. It is also very useful to realize that

$$\widehat{D}_i(1 + \hat{r}_i) = (1 + \hat{r}_i), \tag{68}$$

so that $\widehat{D}_i$ does not change expressions that are $\hat{r}_i$-invariant. Using this fact can reduce computations significantly.

The Demazure character formula can be written simply as

$$\mathrm{ch}_\lambda(a) = \widehat{D}_L(a^\lambda), \tag{69}$$

where we have written $\widehat{D}_L := \widehat{D}_{w_L}$ for short. Equivalently, we can write

$$\widehat{\mathrm{ch}} = \widehat{D}_L, \tag{70}$$

for the operator $\widehat{\mathrm{ch}}$ introduced in (20).

As an example, consider the $su(3)$ representation of highest weight $\lambda = 2\Lambda^1 + \Lambda^2$. We will use the reduced decomposition $\widehat{D}_L = \widehat{D}_1\widehat{D}_2\widehat{D}_1$. First,

$$\widehat{D}_1 a^{2\Lambda^1+\Lambda^2} = a^{2\Lambda^1+\Lambda^2} + a^{2\Lambda^2} + a^{-2\Lambda^1+3\Lambda^2}. \tag{71}$$

Then

$$\begin{aligned}\widehat{D}_2\widehat{D}_1 a^{2\Lambda^1+\Lambda^2} &= \left(a^{2\Lambda^1+\Lambda^2} + a^{3\Lambda^1-\Lambda^2}\right) + \left(a^{2\Lambda^2} + a^{\Lambda^1} + a^{2\Lambda^1-2\Lambda^2}\right)\\ &\quad + \left(a^{-2\Lambda^1+3\Lambda^2} + a^{-\Lambda^1+\Lambda^2} + a^{-\Lambda^2} + a^{\Lambda^1-3\Lambda^2}\right).\end{aligned} \tag{72}$$

To avoid generating terms with negative integer coefficients, that will eventually cancel anyway, we separate out the $\hat{r}_1$-invariant part of this result,

$$\left(a^{2\Lambda^1+\Lambda^2} + a^{2\Lambda^2} + a^{-2\Lambda^1+3\Lambda^2}\right) + \left(a^{\Lambda^1} + a^{-\Lambda^1+\Lambda^2}\right) + \left(a^{-\Lambda^2}\right) \tag{73}$$

before applying $\widehat{D}_1$. By virtue of (68), we then need only compute

$$\begin{aligned}\widehat{D}_1\left(a^{3\Lambda^1-\Lambda^2} + a^{2\Lambda^1-2\Lambda^2} + a^{\Lambda^1-3\Lambda^2}\right) &= \left(a^{3\Lambda^1-\Lambda^2} + a^{\Lambda^1} + a^{-\Lambda^1+\Lambda^2} + a^{-3\Lambda^1+2\Lambda^2}\right)\\ &\quad + \left(a^{2\Lambda^1-2\Lambda^2} + a^{-\Lambda^2} + a^{-2\Lambda^1}\right) + \left(a^{\Lambda^1-3\Lambda^2} + a^{-\Lambda^1-2\Lambda^2}\right).\end{aligned} \tag{74}$$

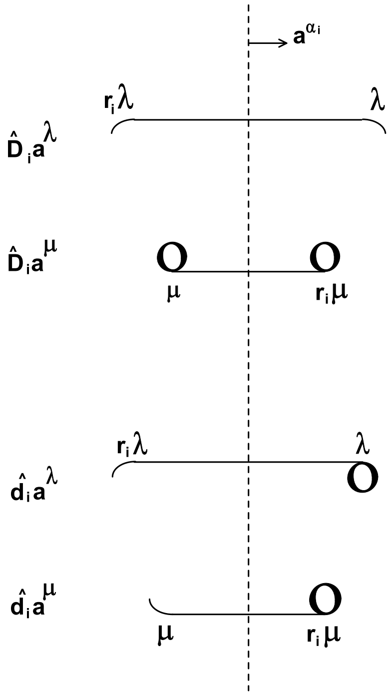


**Figure 1.** Action of the Demazure operators $\widehat{D}_i$ and $\hat{d}_i$. The weight $\lambda$ has positive, integer Dynkin label $\lambda_i$, while $\mu_i$ is a negative integer.

Adding this last result to (73) then gives the character $\mathrm{ch}_{2\Lambda^1+\Lambda^2}(a)$, without the need for cancellations between positive and negative terms, as in the Weyl character formula.

Also useful are operators $\hat{d}_i$, defined by

$$\widehat{D}_i =: 1 + \hat{d}_i, \qquad \hat{d}_i := (1 - a^{-\alpha_i})^{-1} a^{-\alpha_i} (1 - \hat{r}_i). \tag{75}$$

Their action is

$$\hat{d}_i(a^\phi) = \begin{cases} a^{\phi-\alpha_i} + a^{\phi-2\alpha_i} + \cdots + a^{\phi-\phi_i\alpha_i}, & \phi_i \geqslant 1; \\ 0, & \phi_i = 0; \\ -a^\phi - a^{\phi+\alpha_i} - \cdots - a^{\phi+(|\phi_i|-1)\alpha_i}, & \phi_i \leqslant -1. \end{cases} \tag{76}$$

Note that the number of terms in all three of these last expressions is $|\phi_i|$.

Cartoons of the actions of the Demazure operators are given in figure 1. They make clear certain relations, such as $\hat{r}_i\widehat{D}_i = \widehat{D}_i$, $\widehat{D}_i = \hat{d}_i + 1$, $\hat{r}_i\hat{d}_i = a^{\alpha_i}\hat{d}_i$, $\hat{d}_i\hat{r}_i = -\hat{d}_i$, etc. The vertical, dashed line in the figure represents the hyperplane in weight space where the $i$th Dynkin label vanishes. The actions are indicated both for a weight $\lambda$, with positive Dynkin label $\lambda_i$, and a weight $\mu$, with $\mu_i < 0$. Raised, horizontal lines represent strings of terms like $a^\lambda + a^{\lambda-\alpha_i} + \cdots + a^{r_i\lambda}$, with positive coefficients +1. Lowered, horizontal lines correspond to such strings with $-1$ as their coefficients. The circles, consisting as they do of a raised and a lowered part, contribute 0, but emphasize that there is no term $a^\lambda$ in $\hat{d}_i a^\lambda$, e.g.

A unique operator $\hat{d}_w$ can again be defined for any $w \in W$, using reduced decompositions of $w$, if we set

$$\hat{d}_{id}(a^\lambda) = a^\lambda. \tag{77}$$

In agreement with (67), we have

$$\hat{d}_i^2 + \hat{d}_i = 0, \tag{78}$$

so that the Demazure character formula (69) can be rewritten as

$$\mathrm{ch}_\lambda(a) = \sum_{w\in W} \hat{d}_w(a^\lambda), \tag{79}$$

or

$$\widehat{\mathrm{ch}} = \sum_{w\in W} \hat{d}_w. \tag{80}$$

We will now apply the Demazure character formulae to the calculation of character generators, following [11]. In his remarkable paper, Gaskell discovered some of the Demazure results on characters independently, and applied them to character generators. The motivation was to find formulae that did not involve negative terms and cancellations, such as the general one (11) due to Patera and Sharp [21]. The relevant minus signs can be traced to the det $w$ factor in the Weyl character formula (9). As illustrated by the $A_2 \cong su(3)$ example above, however, the Demazure character formula can avoid such negative terms, and so can lead to more useful formulae for X.

To save writing, let us introduce the notation

$$\lfloor x\rfloor := (1-x)^{-1} = \sum_{n=0}^{\infty} x^n. \tag{81}$$

The generating function for highest weights can be written as

$$\mathrm{H}(L,a) := \prod_{\Lambda\in F}(1-L^\Lambda a^\Lambda)^{-1} = \prod_{\Lambda\in F}\lfloor L^\Lambda a^\Lambda\rfloor, \tag{82}$$

and the generating function of interest is then

$$\mathrm{X} = \widehat{\mathrm{ch}}\left(\prod_{\Lambda\in F}(1-L^\Lambda a^\Lambda)^{-1}\right) = \widehat{\mathrm{ch}}(\mathrm{H}) = \widehat{D}_L(\mathrm{H}). \tag{83}$$

This general formula for the character generator was first written by Gaskell [11]. Choosing a reduced decomposition of $\widehat{D}_L$, X can be calculated by successive applications of the basic Demazure operators $\widehat{D}_i$.

To proceed, Gaskell [11] derived the product rule

$$\widehat{D}_i(FG) = (\widehat{D}_iF)G + (\hat{r}_iF)(\hat{d}_iG). \tag{84}$$

This also implies

$$\widehat{D}_i(FG) = F(\widehat{D}_iG) + (\hat{d}_iF)(\hat{r}_iG). \tag{85}$$

In terms of the operators $\hat{d}_i$, these identities read as

$$\hat{d}_i(FG) = (\hat{d}_iF)G + (\hat{r}_iF)(\hat{d}_iG) \tag{86}$$

and

$$\hat{d}_i(FG) = F(\hat{d}_iG) + (\hat{d}_iF)(\hat{r}_iG). \tag{87}$$

For us, the most useful of these product rules will be (85).

To apply the Demazure operators on individual factors of $\mathrm{H}(L,a)$ and the results, we need

$$\hat{d}_i\lfloor F\rfloor = \lfloor F\rfloor \hat{d}_iF\lfloor \hat{r}_iF\rfloor \tag{88}$$

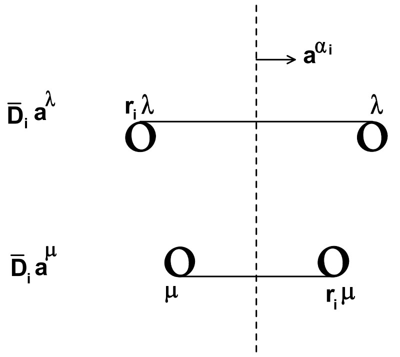


**Figure 2.** The action of the operator $\bar{D}_i = \widehat{D}_i a^{-\alpha_i}$.

or

$$\widehat{D}_i \lfloor F \rfloor = \lfloor F \rfloor + \lfloor F \rfloor \hat{d}_i F \lfloor \hat{r}_i F \rfloor. \tag{89}$$

$\widehat{D}_i$ always acts to produce an $r_i$-invariant expression. The right-hand side of the last result is therefore $r_i$-invariant, although it is not obvious. A manifestly invariant formula can be written, however, as

$$\widehat{D}_i \lfloor F \rfloor = \lfloor F \rfloor (1 + \bar{D}_i F) \lfloor \hat{r}_i F \rfloor. \tag{90}$$

Since we will need to write it often, we have defined

$$\bar{D}_i := \hat{d}_i - \hat{r}_i. \tag{91}$$

From the expression (63), we can show that

$$\bar{D}_i = \widehat{D}_i a^{-\alpha_i}, \tag{92}$$

demonstrating that $\bar{D}_i$, too, generates $r_i$-invariant expressions. The action of $\bar{D}_i$ is depicted in figure 2.

Before treating examples, let us use Demazure operators to re-derive the new formula $\mathrm{X} = \mathrm{Y}/\mathrm{Z}$, with numerator and denominator given by (22) and (12), respectively. Re-write (83) as

$$\mathrm{X} = \widehat{D}_L \left( \frac{\mathcal{Y} H}{\mathcal{Y}} \right) = \widehat{D}_L (\mathrm{Z}^{-1} \mathcal{Y}). \tag{93}$$

Since Z is $W$-invariant, it is annihilated by all the $\hat{d}_i$. The product rule (85) therefore yields

$$\mathrm{X} = \mathrm{Z}^{-1} \widehat{D}_L (\mathcal{Y}) = \mathrm{Z}^{-1} \widehat{\mathrm{ch}} (\mathcal{Y}), \tag{94}$$

the desired result.

### *4.1. Rank-two simple Lie algebras*

The calculations quickly become unwieldy with increasing rank. For simplicity, we will restrict to consideration of the simple Lie algebras of rank two. We believe our results are indicative of general properties of the character generators of the simple Lie algebras, however.

We should point out that we make certain choices when we perform our Demazure calculations, such as the order of factors in the highest-weight generating function H, the reduced decomposition of $w_L$, and which of the product rules (84)–(85) we apply. Of course, none of these choices changes the final result, but they can simplify the calculations substantially, and affect the way the final answer is expressed. It will become clear in section 5 why we make the choices we do, when a connection with graphs is established.

*4.1.1. $A_2$.* Choosing the reduced decomposition $w_L = r_2r_1r_2$, we need to calculate

$$\mathrm{X} = \widehat{D}_L\mathrm{H} = \widehat{D}_2\widehat{D}_1\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor). \tag{95}$$

First consider the application of $\widehat{D}_2$. Since $\lfloor L_1a_1\rfloor$ is $\hat{r}_2$-invariant, it is unaffected. So,

$$\begin{aligned}\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor) &= \lfloor L_2a_2\rfloor(1+\bar{D}_2L_2a_2)\big\lfloor L_2a_1a_2^{-1}\big\rfloor\lfloor L_1a_1\rfloor\\ &= \lfloor L_2a_2\rfloor\big\lfloor L_2a_1a_2^{-1}\big\rfloor\lfloor L_1a_1\rfloor,\end{aligned} \tag{96}$$

where we used (90), and $\bar{D}_2L_2a_2 = 0$.

When we apply $\widehat{D}_1$, $\lfloor L_2a_2\rfloor$ is untouched: $\widehat{D}_1\lfloor L_2a_2\rfloor = \lfloor L_2a_2\rfloor$. Using the product rule (85), we get

$$\begin{aligned}\widehat{D}_1(\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor)) &= \underline{\lfloor L_2a_2\rfloor}\big\lfloor L_2a_1a_2^{-1}\big\rfloor\lfloor L_1a_1\rfloor(1+\bar{D}_1L_1a_1)\big\lfloor L_1a_1^{-1}a_2\big\rfloor\\ &\quad+\underline{\lfloor L_2a_2\rfloor}\big\lfloor L_2a_1a_2^{-1}\big\rfloor\hat{d}_1L_2a_1a_2^{-1}\big\lfloor L_2a_1^{-1}\big\rfloor\big\lfloor L_1a_1^{-1}a_2\big\rfloor,\end{aligned} \tag{97}$$

which simplifies since $\bar{D}_1L_1a_1 = 0$. At this point, we can save effort by anticipating the application of $\widehat{D}_2$, and rewriting the result as

$$\begin{aligned}\widehat{D}_1(\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor)) &= \underline{\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor)}\big\lfloor L_1a_1^{-1}a_2\big\rfloor\\ &\quad+\underline{\lfloor L_2a_2\rfloor\big\lfloor L_2a_1a_2^{-1}\big\rfloor L_2a_1^{-1}\big\lfloor L_2a_1^{-1}\big\rfloor}\big\lfloor L_1a_1^{-1}a_2\big\rfloor,\end{aligned} \tag{98}$$

since $\hat{d}_1L_2a_1a_2^{-1} = L_2a_1^{-1}$. In this last expression, terms that are $\hat{r}_2$-invariant are made plain by underlines, and all such terms are annihilated by $\widehat{D}_2$. This procedure saves considerable work in more complicated cases.

For $A_2$, we therefore find

$$\begin{aligned}\widehat{D}_2(\widehat{D}_1(\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor))) &= \underline{\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor)}\big\lfloor L_1a_1^{-1}a_2\big\rfloor\big(1+\bar{D}_2L_1a_1^{-1}a_2\big)\big\lfloor L_1a_2^{-1}\big\rfloor\\ &\quad+\underline{\lfloor L_2a_2\rfloor\big\lfloor L_2a_1a_2^{-1}\big\rfloor L_2a_1^{-1}\big\lfloor L_2a_1^{-1}\big\rfloor}\big\lfloor L_1a_1^{-1}a_2\big\rfloor\\ &\quad\times\big(1+\bar{D}_2L_1a_1^{-1}a_2\big)\big\lfloor L_1a_2^{-1}\big\rfloor.\end{aligned} \tag{99}$$

Since $\bar{D}_2L_1a_1^{-1}a_2$ vanishes, the final result is

$$\begin{aligned}\mathrm{X} &= \widehat{D}_2\widehat{D}_1\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor)\\ &= \lfloor L_2a_2\rfloor\big\lfloor L_2a_1a_2^{-1}\big\rfloor\lfloor L_1a_1\rfloor\big\lfloor L_1a_1^{-1}a_2\big\rfloor\big\lfloor L_1a_2^{-1}\big\rfloor\\ &\quad+\lfloor L_2a_2\rfloor\big\lfloor L_2a_1a_2^{-1}\big\rfloor L_2a_1^{-1}\big\lfloor L_2a_1^{-1}\big\rfloor\big\lfloor L_1a_1^{-1}a_2\big\rfloor\big\lfloor L_1a_2^{-1}\big\rfloor.\end{aligned} \tag{100}$$

Incidentally, the same result can be found by

$$\begin{aligned}\mathrm{Y} &= \widehat{D}_L\mathcal{Y} = \widehat{D}_1\widehat{D}_2\widehat{D}_1\mathcal{Y}\\ &= \widehat{D}_1\widehat{D}_2\big\{\big(1-L_2a_1a_2^{-1}\big)\big(1-L_2a_1^{-1}\big)\big[\widehat{D}_1\big(1-L_1a_1^{-1}a_2\big)\big]\big(1-L_2a_2^{-1}\big)\big\}\\ &= \widehat{D}_1\widehat{D}_2\big\{\big(1-L_2a_1a_2^{-1}\big)\big(1-L_2a_1^{-1}\big)\big(1-L_2a_2^{-1}\big)\big\}\\ &= \widehat{D}_1\big\{\big(1-L_2a_1^{-1}\big)(1-L_1a_2)^{-1}+(1-L_2a_2)\big(1-L_2a_1^{-1}\big)L_1a_2^{-1}\big\}\\ &= \big(1-L_1a_2^{-1}\big)+(1-L_2a_2)L_1a_2^{-1},\end{aligned} \tag{101}$$

for example.

The structure of the generating functions is more easily seen if we write

$$\begin{aligned}&A = L_2a_2, \qquad B = L_2a_1a_2^{-1}, \qquad C = L_2a_1^{-1},\\ &D = L_1a_1, \qquad E = L_1a_1^{-1}a_2, \qquad F = L_1a_2^{-1},\end{aligned} \tag{102}$$

so that

$$\mathrm{X} = \lfloor A\rfloor\lfloor B\rfloor(\lfloor D\rfloor + C\lfloor C\rfloor)\lfloor E\rfloor\lfloor F\rfloor. \tag{103}$$

First, reconsider the three choices of incompatible product displayed in (56). They are $DC$, $EB$ and $FA$, respectively. Expanding the expression of (103) using (81) results in no terms involving the product $DC$. Choosing $DC$ as the sole incompatible product for the $A_2$ case therefore leads to (103). Rewriting that expression as $Z^{-1}(1-DC)$, where

$$Z^{-1} = \lfloor A\rfloor\lfloor B\rfloor\lfloor C\rfloor\lfloor D\rfloor\lfloor E\rfloor\lfloor F\rfloor, \tag{104}$$

makes clear that the incompatibility $DC$ is related to (103). Similarly, we can write

$$\begin{aligned} \mathrm{X} &= \lfloor C\rfloor\lfloor A\rfloor(\lfloor E\rfloor + B\lfloor B\rfloor)\lfloor F\rfloor\lfloor D\rfloor = Z^{-1}(1-EB) \\ &= \lfloor B\rfloor\lfloor C\rfloor(\lfloor F\rfloor + A\lfloor A\rfloor)\lfloor D\rfloor\lfloor E\rfloor = Z^{-1}(1-FA), \end{aligned} \tag{105}$$

corresponding to the other two choices $EB$ and $FA$, respectively, for the incompatible product.

*4.1.2. $B_2$.* The longest element of the $B_2$ Weyl group has the reduced decompositions $w_L = r_1r_2r_1r_2 = r_2r_1r_2r_1$. Choosing the first, we write

$$\mathrm{X} = \widehat{D}_L\mathrm{H} = \widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor), \tag{106}$$

and perform the calculations in the same manner as the $A_2$ ones. We get

$$\begin{aligned} \widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor) &= \lfloor L_2a_2\rfloor\left\lfloor L_2a_1a_2^{-1}\right\rfloor\lfloor L_1a_1\rfloor\left\lfloor L_1a_1^{-1}a_2^2\right\rfloor \\ &\quad\times\left(1+\bar{D}_2L_1a_1^{-1}a_2^2\right)\left\lfloor L_1a_1a_2^{-2}\right\rfloor\left\lfloor L_1a_1^{-1}\right\rfloor \\ &\quad+\lfloor L_2a_2\rfloor\left\lfloor L_2a_1a_2^{-1}\right\rfloor L_2a_1^{-1}a_2\left\lfloor L_2a_1^{-1}a_2\right\rfloor\left\lfloor L_1a_1^{-1}a_2^2\right\rfloor \\ &\quad\times\left(1+\bar{D}_2L_1a_1^{-1}a_2^2\right)\left\lfloor L_1a_1a_2^{-2}\right\rfloor\left\lfloor L_1a_1^{-1}\right\rfloor \\ &\quad+\lfloor L_2a_2\rfloor\left\lfloor L_2a_1a_2^{-1}\right\rfloor\left\lfloor L_2a_1^{-1}a_2\right\rfloor L_2a_2^{-1}\left\lfloor L_2a_2^{-1}\right\rfloor \\ &\quad\times\left\lfloor L_1a_1a_2^{-2}\right\rfloor\left\lfloor L_1a_1^{-1}\right\rfloor. \end{aligned} \tag{107}$$

Using the notation

$$\begin{aligned} &A = L_2a_2, && B = L_2a_1a_2^{-1}, && C = L_2a_1^{-1}a_2, && D = L_2a_2^{-1}, \\ &E = L_1a_1, && F = L_1a_1^{-1}a_2^2, && G = L_1a_1a_2^{-2}, && H = L_1a_1^{-1}, \end{aligned} \tag{108}$$

the $B_2$ character generator takes a compact form. Defining

$$\lfloor AB\rfloor := \lfloor A\rfloor\lfloor B\rfloor, \tag{109}$$

and similarly for more than two factors, we can write

$$\mathrm{X} = \lfloor ABC\rfloor D\lfloor DGH\rfloor + \lfloor AB\rfloor C\lfloor CF\rfloor(1+z)\lfloor GH\rfloor + \lfloor ABEF\rfloor(1+z)\lfloor GH\rfloor. \tag{110}$$

Here we have also defined $z := \bar{D}_2L_1a_1^{-1}a_2^2 = L_1$ for the sole inside generator required for the $B_2$ generating function.

The weights of the generators $A$ through $H$ and $z$ are depicted in figure 3. For all character generators, the generator weights fill out the $r$ fundamental weight diagrams of the relevant rank-$r$ simple Lie algebra. The two fundamental weight diagrams of $B_2$ are shown in the figure.

The form of the character generator can be related to the set of incompatible products obtained above. In terms of the integrity basis elements, the choice (59) gives $\{EC, ED, FD, zD\}$ as the set of incompatible products. It is easily seen that the expression (110) does not contain these products, but does contain all other products quadratic in the elements of $\{A, B, \ldots, F, z\}$.

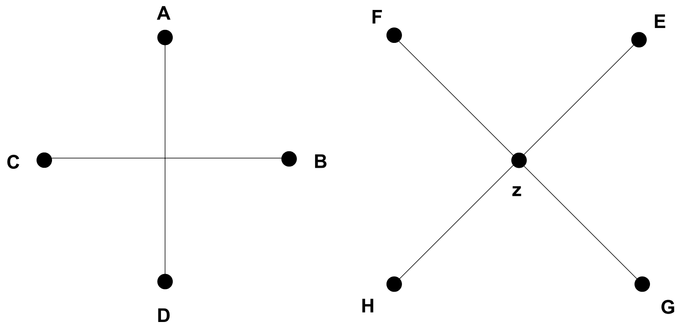


**Figure 3.** The $B_2$ fundamental weight diagrams. The weights are labelled by the corresponding elements of the integrity basis $I_X$.

*4.1.3. $G_2$.* For $G_2$, $w_L = r_1r_2r_1r_2r_1r_2 = r_2r_1r_2r_1r_2r_1$ are the two reduced decompositions of the longest element of the Weyl group. We consider

$$\mathrm{X} = \widehat{D}_L\mathrm{H} = \widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2(\lfloor L_2a_2\rfloor\lfloor L_1a_1\rfloor) \tag{111}$$

here. After a lengthy calculation, we find

$$\begin{aligned} X = \widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2(\lfloor A\rfloor\lfloor G\rfloor) \\ = \lfloor A\rfloor\lfloor B\rfloor\lfloor G\rfloor\lfloor H\rfloor(1+\bar{D}_2H)\lfloor I\rfloor(1+\bar{D}_1I)\lfloor J\rfloor(1+\bar{D}_2J)\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor C\lfloor C\rfloor\lfloor H\rfloor(1+\bar{D}_2H)\lfloor I\rfloor(1+\bar{D}_1I)\lfloor J\rfloor(1+\bar{D}_2J)\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor\lfloor C\rfloor\hat{d}_2C\lfloor D\rfloor\lfloor I\rfloor(1+\bar{D}_1I)\lfloor J\rfloor(1+\bar{D}_2J)\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor\lfloor G\rfloor\lfloor H\rfloor\hat{d}_1\bar{D}_2H\lfloor J\rfloor(1+\bar{D}_2J)\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor C\lfloor C\rfloor\lfloor H\rfloor\hat{d}_1\bar{D}_2H\lfloor J\rfloor(1+\bar{D}_2J)\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor\lfloor C\rfloor(1+\bar{D}_2C)\lfloor D\rfloor E\lfloor E\rfloor\lfloor J\rfloor(1+\bar{D}_2J)\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor\lfloor G\rfloor\lfloor H\rfloor\bar{D}_2\hat{d}_1\bar{D}_2H\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor C\lfloor C\rfloor\lfloor H\rfloor\bar{D}_2\hat{d}_1\bar{D}_2H\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor\lfloor G\rfloor\lfloor H\rfloor(1+\bar{D}_2H)\lfloor I\rfloor\hat{r}_2\hat{d}_1\bar{D}_2H\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor C\lfloor C\rfloor\lfloor H\rfloor(1+\bar{D}_2H)\lfloor I\rfloor\hat{r}_2\hat{d}_1\bar{D}_2H\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor\lfloor C\rfloor\hat{d}_2C\lfloor D\rfloor\lfloor I\rfloor\hat{r}_2\hat{d}_1\bar{D}_2H\lfloor K\rfloor\lfloor L\rfloor \\ +\lfloor A\rfloor\lfloor B\rfloor\lfloor C\rfloor(1+\bar{D}_2C)\lfloor D\rfloor\lfloor E\rfloor F\lfloor F\rfloor\lfloor K\rfloor\lfloor L\rfloor, \end{aligned} \tag{112}$$

where we have used the following notation for the $G_2$ outside generators:

$$\begin{array}{lll} A = L_2a_2, & B = L_2a_1a_2^{-1}, & C = L_2a_1^{-1}a_2^2, \\ D = L_2a_1a_2^{-2}, & E = L_2a_1^{-1}a_2, & F = L_2a_2^{-1}, \\ G = L_1a_1, & H = L_1a_1^{-1}a_2^3, & I = L_1a_1^2a_2^{-3}, \\ J = L_1a_1^{-2}a_2^3, & K = L_1a_1a_2^{-3}, & L = L_1a_1^{-1}. \end{array} \tag{113}$$

The weights of these outside elements (see (15)) of the integrity basis are depicted in figure 4, where the weight diagrams of the fundamental representations of $G_2$ are drawn. The other weights are labelled by our notation for the corresponding inside generators.

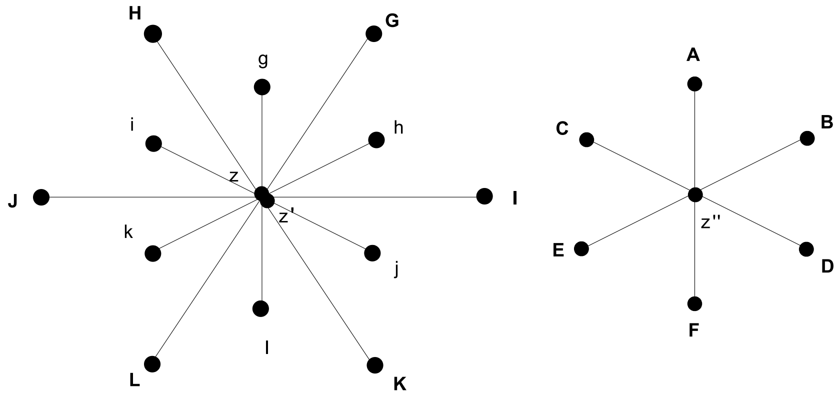


**Figure 4.** The $G_2$ fundamental weight diagrams. The weights are indicated by the elements of $I_X$.

The inside generators, the elements of $I_{\text{in}}$, make their appearance when we calculate numerator factors, such as $\hat{r}_2\hat{d}_1\bar{D}_2H = j$. The final result is therefore

$$\begin{aligned} X = {} & \lfloor ABGH\rfloor(1+g+h)\lfloor I\rfloor(1+z')\lfloor J\rfloor(1+k+\ell)\lfloor KL\rfloor \\ & + \lfloor AB\rfloor C\lfloor CH\rfloor(1+g+h)\lfloor I\rfloor(1+z')\lfloor J\rfloor(1+k+\ell)\lfloor KL\rfloor \\ & + \lfloor ABC\rfloor(z''+D)\lfloor DI\rfloor(1+z')\lfloor J\rfloor(1+k+\ell)\lfloor KL\rfloor \\ & + \lfloor ABGH\rfloor i\lfloor J\rfloor(1+k+\ell)\lfloor KL\rfloor \\ & + \lfloor AB\rfloor C\lfloor CH\rfloor i\lfloor J\rfloor(1+k+\ell)\lfloor KL\rfloor \\ & + \lfloor ABC\rfloor(1+z'')\lfloor D\rfloor E\lfloor EJ\rfloor(1+k+\ell)\lfloor KL\rfloor \\ & + \lfloor ABGH\rfloor z\lfloor KL\rfloor + \lfloor AB\rfloor C\lfloor CH\rfloor z\lfloor KL\rfloor \\ & + \lfloor ABGH\rfloor(1+g+h)\lfloor I\rfloor j\lfloor KL\rfloor \\ & + \lfloor AB\rfloor C\lfloor CH\rfloor(1+g+h)\lfloor I\rfloor j\lfloor KL\rfloor \\ & + \lfloor ABC\rfloor(z''+D)\lfloor DI\rfloor j\lfloor KL\rfloor \\ & + \lfloor ABC\rfloor(1+z'')\lfloor DE\rfloor F\lfloor FKL\rfloor. \end{aligned} \tag{114}$$

Here we have again shortened by using $\lfloor A\rfloor\lfloor B\rfloor =: \lfloor AB\rfloor$, etc.

Now consider the choice of incompatible products underlying this expression for X written in terms of integrity basis elements. By inspecting (114), we can find the incompatibilities between outer generators:

$$\{GC, GD, GE, GF, HD, HE, HF, IE, IF, JF\}. \tag{115}$$

None of these products appears in the expansion of any of the terms of (114). Therefore, we write

$$\mathrm{Y}^{(2)}_{\text{out,out}} = G(C+D+E+F) + H(D+E+F) + I(E+F) + JF. \tag{116}$$

Similarly, for inner· outer incompatible products, we find

$$\begin{aligned} \mathrm{Y}^{(2)}_{\text{in,out}} = {} & z''(G+H) + (z+g+h+i)(D+E+F) + z'(E+F) \\ & + j(E+F) + (k+\ell)F + z(I+J) + iI + jJ. \end{aligned} \tag{117}$$

Compatible inner· inner products give

$$\mathrm{Y}^{(2)}_{\text{in,in}} = (z''+g+h)(j+k+\ell+z') + i(k+\ell+z') + z'(k+\ell). \tag{118}$$

Substituting these last three results into (61) verifies the result (60) derived from the general formula (22).

## 5. Character generators, semi-standard tableaux, posets and graphs

By (2), the character generator is the generating function of the multiplicities $\mathrm{mult}_\lambda(\sigma)$:

$$X = \sum_{\lambda\in P_\geqslant}\sum_{\sigma\in P} L^\lambda a^\sigma \mathrm{mult}_\lambda(\sigma). \tag{119}$$

Many combinatorial ways of calculating such multiplicities are known, including those involving Young tableaux and variants. This 'microscopic' point of view leads to an improved understanding of the structure of the character generators. The first to exploit this fact was Stanley [23], for the algebras $A_r \cong su(r+1)$. King [15] extended Stanley's work to include the algebras $C_r \cong \mathrm{sp}(2r)$.

Most relevant to us, however, was the connection made explicit by Baclawski [1] to certain partially-ordered sets, or posets, related to tableaux. A poset $\mathcal{P}$ is a set, together with a binary operation (partial order) $\geqslant$, satisfying reflexivity ($x \geqslant x, \forall x \in \mathcal{P}$), antisymmetry (if $x \geqslant y$ and $y \geqslant x$, then $x = y$) and transitivity (if $x \geqslant y$ and $y \geqslant z$, then $x \geqslant z$). It is a partial order because two elements $x, y \in \mathcal{P}$ can be incomparable, i.e. neither $x \geqslant y$ nor $y \geqslant x$ is true.

The connection with posets was already made in [23], but much less directly than in [1]. Baclawski emphasized its importance and wrote explicit formulae in terms of poset objects. Later these considerations were generalized to all the classical Lie algebras $A_r, B_r, C_r, D_r$ (or all $su(N), so(N), sp(2N)$) in [16, 17], using generalized Young tableaux[5]. At about the same time, Baclawski and Towber [2] treated the exceptional $G_2$ algebra by introducing a generalization of a poset.

In the remainder of this section, we will treat the algebras $A_r$, $B_2$ and $G_2$ in turn. This first case is the simplest, and best understood.

### *5.1.* $A_r \cong su(r+1)$

Certain posets are encoded in the structure of Young tableaux, and related objects. For example, consider the algebras $A_r \cong su(r+1)$. Their multiplicities $\mathrm{mult}_\lambda(\sigma)$ equal the number of semi-standard Young tableaux of shape $\lambda$ and weight $\sigma$ (see [8], e.g.). These Young tableaux can be constructed by joining together the semi-standard tableaux of the fundamental representation, and these fundamental tableaux become the columns of the full semi-standard tableaux. The only complication is that they must be placed in a certain order.

More precisely, the columns of the semi-standard tableaux, the fundamental tableaux, are the elements of a poset $\mathcal{P}$. The partial order can be encoded in a so-called Hasse diagram, a graph whose vertices are the elements of the poset, and whose edges indicate the order (see [24], e.g.). The poset $\mathcal{P}$ is locally finite, meaning it has an order that is completely determined by its cover relations. $x > y$ is a cover relation if no poset element $z$ exists such that $x > z > y$. To every cover relation $x > y$ of the poset $\mathcal{P}$, there is an edge $\{x, y\}$ in its Hasse diagram $\mathcal{H}(\mathcal{P})$.

The Hasse diagram relevant to $su(3)$ semi-standard tableaux is drawn in figure 5, with the fundamental semi-standard tableaux drawn where the corresponding vertices would be. They are lined up horizontally, to make obvious the connection with the semi-standard tableaux for $su(3)$. Any number of copies of the fundamental tableaux of each kind can be used to build a valid semi-standard tableaux, as long as the partial order is respected.

Recall that the weight of an integrity basis element, $L^\Lambda a^\mu$, is $\mu$ (while the fundamental weight $\Lambda$ is its shape). The weights of the fundamental semi-standard tableaux are the weights of the integrity basis elements (102) for the character generator. The Hasse diagram can be

[5] For a discussion of character generators and tableaux methods with a different emphasis, see [5].

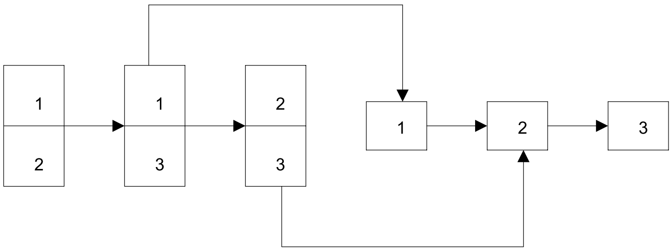


**Figure 5.** $su(3)$ Hasse diagram with the fundamental semi-standard tableaux.

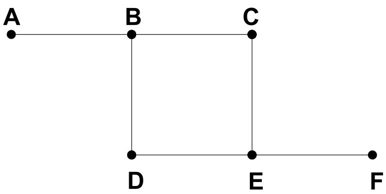


**Figure 6.** Hasse diagram of the $su(3)$ fundamental poset.

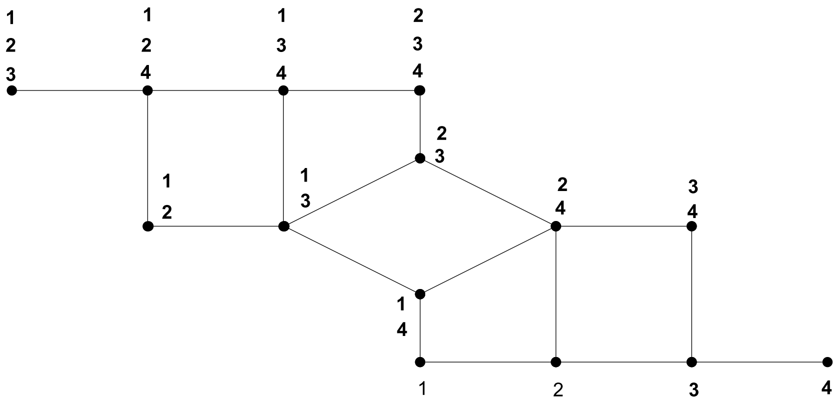


**Figure 7.** Hasse diagram of the $su(4)$ fundamental poset, with the fundamental tableaux indicated next to the corresponding vertices.

labelled by those basis elements, and then the diagram provides a method of constructing the generating function directly. For $su(3)$, the resulting Hasse diagram is drawn in figure 6. The corresponding $su(4) \cong A_3$ Hasse diagram is shown in figure 7.

Consider the $su(3)$ character generator (103). The two terms are easily seen to correspond to the two longest paths (or walks) $ABDEF$ and $ABCEF$ on the Hasse diagram from 'the beginning' $A$, or the greatest element, to 'the end', or least element, $F$. These two paths correspond to the two maximal chains, or totally ordered sets, $A \geqslant B \geqslant D \geqslant E \geqslant F$ and $A \geqslant B \geqslant C \geqslant E \geqslant F$ in the corresponding poset. The two maximal chains are treated in equal fashion, since

$$\lfloor D \rfloor + C \lfloor C \rfloor = 1 + D \lfloor D \rfloor + C \lfloor C \rfloor. \tag{120}$$

From this last expression, one can see that the extra factor of $C$ in (103) is necessary to avoid over-counting.

The point of view just explained was discovered by Baclawski [1], and applied to the simple algebras $A_r$ and $C_r$ (as well as to $U(N)$). Using generalized tableaux, character generators for all classical algebras ($A_r$, $B_r$, $C_r$, and $D_r$) were studied in [16, 17].

Let us write Baclawski's results, concentrating on the case of $A_r$. Denote by $\mathcal{P}$ the poset with fundamental tableaux as elements, the so-called fundamental poset. We can label the elements of the poset with the corresponding elements of the integrity basis $I_{\mathrm{X}}$, as in the diagram of figure 6 for $su(3)$. The result is called a labelling of the poset $\mathcal{P}$, since the labels can be added and multiplied. A multi-chain is a chain with repeated elements, such as $m = A \geqslant A \geqslant B \geqslant D \geqslant D \geqslant E \geqslant F \geqslant F \geqslant F$, where, by abuse of notation, we use the labels to denote the poset elements. The label of such a multi-chain is easily obtained:

$$\ell(m) = \ell(A \geqslant A \geqslant B \geqslant D \geqslant D \geqslant E \geqslant F \geqslant F \geqslant F) = A^2 B D^2 F^3. \qquad (121)$$

The first result is simply written as

$$\mathrm{X} = \sum_{m \in M(\mathcal{P})} \ell(m), \qquad (122)$$

where $M(\mathcal{P})$ denotes the set of multi-chains of $\mathcal{P}$.

As pointed out above, the relevance to X of maximal chains is immediately obvious. To write the formula [1] that makes the connection explicit, consider the poset $\hat{\mathcal{P}}$, the extended fundamental poset, obtained by adjoining two new elements, $\hat{0}$ and $\hat{1}$, to the poset $\mathcal{P}$. The element $\hat{0}$ satisfies $x \geqslant \hat{0}$, and $\hat{1}$ obeys $\hat{1} \geqslant x$, both for all $x \in \hat{\mathcal{P}}$. The labelling of $\hat{\mathcal{P}}$ that we use is simply obtained by adjoining the labels $\ell(\hat{0}) = \ell(\hat{1}) = 1$ to the labelling of $\mathcal{P}$.

The links of a poset are relevant here. A chain $\mathcal{C}$ of a poset $\mathcal{P}$ is called saturated if no $z \in \mathcal{P}\backslash\mathcal{C}$ exists such that $x \geqslant z \geqslant y$ for $x, y \in \mathcal{C}$, such that $\mathcal{C} \cup \{z\}$ is a chain. Roughly speaking, there are no gaps in a saturated chain. A cover relation is a two-element saturated chain, and a link is a saturated chain with three-elements.

Let $\mathrm{Link}(\hat{\mathcal{P}})$ denote the set of links of $\hat{\mathcal{P}}$. For the $su(3)$ case, we have

$$\begin{aligned}\mathrm{Link}(\hat{\mathcal{P}}) = \{&\hat{1} > A > B, A > B > C, A > B > D, B > C > E,\\ &B > D > E, C > E > F, D > E > F, E > F > \hat{0}\}.\end{aligned} \qquad (123)$$

A linking of a poset $\mathcal{P}$ is a partition of $\mathrm{Link}(\hat{\mathcal{P}})$ into two disjoint subsets $\mathrm{Link}^{\pm}(\hat{\mathcal{P}})$, such that, for every pair $x > y$ in $\hat{\mathcal{P}}$, there exists a unique saturated chain $x = x_0 > x_1 > \cdots > x_{n-1} > x_n = y$, every link of which is in $\mathrm{Link}^{+}(\hat{\mathcal{P}})$. For the $A_2$ example, one linking of the poset $\mathcal{P}$ is specified by the choice

$$\mathrm{Link}^{-}(\hat{\mathcal{P}}) = \{B > C > E\}. \qquad (124)$$

Then $\mathrm{Link}^{+}(\hat{\mathcal{P}}) = \mathrm{Link}(\hat{\mathcal{P}})\backslash\mathrm{Link}^{-}(\hat{\mathcal{P}})$.

Another concept required for the formula is that of a descent set $\mathcal{DS}(m)$, of a maximal chain $m = x_0 > x_1 > \cdots > x_n$ of $\hat{\mathcal{P}}$:

$$\mathcal{DS}(m) := \{x_i \mid 0 < i < n \text{ and } (x_{i-1} > x_i > x_{i+1}) \in \mathrm{Link}^{-}(\hat{\mathcal{P}})\}. \qquad (125)$$

Its label is therefore

$$\ell(\mathcal{DS}(m)) = \prod_{x \in \mathcal{DS}(m)} \ell(x). \qquad (126)$$

Let $\mathrm{Max}(\hat{\mathcal{P}})$ denote the set of maximal chains in $\hat{\mathcal{P}}$. Baclawski [1] proved

$$\mathrm{X} = \sum_{m \in \mathrm{Max}(\hat{\mathcal{P}})} \lfloor \ell(m) \rfloor \ell(\mathcal{DS}(m)). \qquad (127)$$

For the extended poset $\hat{\mathcal{P}}$ relevant to $A_2$, $\mathrm{Max}(\hat{\mathcal{P}})$ contains two chains, $\hat{1} > A > B > D > E > F > \hat{0}$ and $\hat{1} > A > B > C > E > F > \hat{0}$. For the first, the linking specified by (124) gives a null descent set, while for the second maximal chain, the descent set is $\{C\}$. The formula (127) therefore immediately reproduces the result (103).

With the alternate choice $\mathrm{Link}^-(\hat{\mathcal{P}}) = \{B > D > E\}$, Baclawski's formula (127) yields $X = \lfloor ABCEF\rfloor + \lfloor AB\rfloor D\lfloor DEF\rfloor$, an equivalent form.

Note that for both linkings of the poset $\mathcal{P}$, the product $CD$ is incompatible. Incompatibilities are fixed by the poset itself, rather than by a choice of linking of $\mathcal{P}$. $CD$ is an incompatible product because $C$ and $D$ are incomparable in the poset, as is clear from its Hasse diagram in figure 6.

We should mention that Baclawski [1] also derived formulae of a recursive nature, that lead to nested expressions for X. Considerably shortened expressions can result this way, since the sub-poset structure is taken into account. Since our goal is an understanding of the full character generator and corresponding (generalized) posets, however, we will not study those formulae here.

In figure 7, the Hasse diagram of the fundamental poset for $A_3 \cong su(4)$ is depicted, with the vertices labelled by the corresponding fundamental semi-standard tableaux. Using (122) or (127) on this diagram yields the $A_3$ character generator in straightforward fashion. Higher ranks involve larger fundamental posets and Hasse diagrams, but do not require any new important complications.

Clearly, the fundamental poset $\mathcal{P}$ encodes the essence of the character generator X, for the algebras $A_r$. This poset can also be constructed without reference to semi-standard tableaux. The alternative construction uses the Weyl group and its Bruhat order (see [14], e.g.). That means it is more easily adaptable to general simple Lie algebras than are the semi-standard tableaux relevant to $A_r \cong su(r+1)$.

The elements of the poset are in one-to-one correspondence with the weights of the $r$ fundamental representations of $A_r$. The poset's cover relations can be stated simply if the vertex corresponding to the weight $\mu$ in $R(\Lambda^j)$ is indicated by the triple $[\Lambda^j, \mu; w]$, $w \in W$. That is, we adjoin a fixed $w \in W$ obeying $\mu = w\Lambda^j$. Of course, there is an ambiguity in the choice of $w$ for a fixed weight $\mu$ of $R(\Lambda^j)$. Consider, however, the reduced decomposition of $w_L$ used in the Demazure calculation of the character generator. For $su(r+1)$, we can use

$$w_L = (r_r r_{r-1} \cdots r_1)(r_r r_{r-1} \cdots r_2)(r_r r_{r-1} \cdots r_3) \cdots (r_r r_{r-1})(r_r). \qquad (128)$$

This expression motivates the label $[\Lambda^r, \Lambda^r; \mathrm{id}]$ and, for the other highest-weight vertices,

$$[\Lambda^j, \Lambda^j; (r_r r_{r-1} \cdots r_{j+1}) \cdots (r_r r_{r-1})(r_r)], \qquad (129)$$

for $j = 1, \ldots, r-1$. Note that the length of the adjoined Weyl elements increases as $j$ decreases in $\Lambda^j$. That is, our choice of reduced decomposition for $w_L$ induces a total order on the set $F$ of fundamental weights.

Once the Weyl elements are fixed for the highest-weight vertices, the Bruhat order can then be used to assign Weyl elements to all the vertices of the required Hasse diagram. The edges of the Hasse diagram are determined by the cover relations:

$$\begin{aligned} [\Lambda^j, \mu; w] \to [\Lambda^i, \nu; v], \qquad & \text{if } i = j \text{ and } w \leftarrow v, \\ & \text{or if } j = i+1 \text{ and } w = v. \end{aligned} \qquad (130)$$

Here $w \leftarrow v$ indicates a cover relation in the Bruhat partial order on $W$. See figure 8 for illustrations of the cases $A_2$ and $A_3$.

Before leaving the $A_r$ algebras, we should mention that the semi-standard tableaux have significance for the vectors (states) in representations, not just their multiplicities. As is well

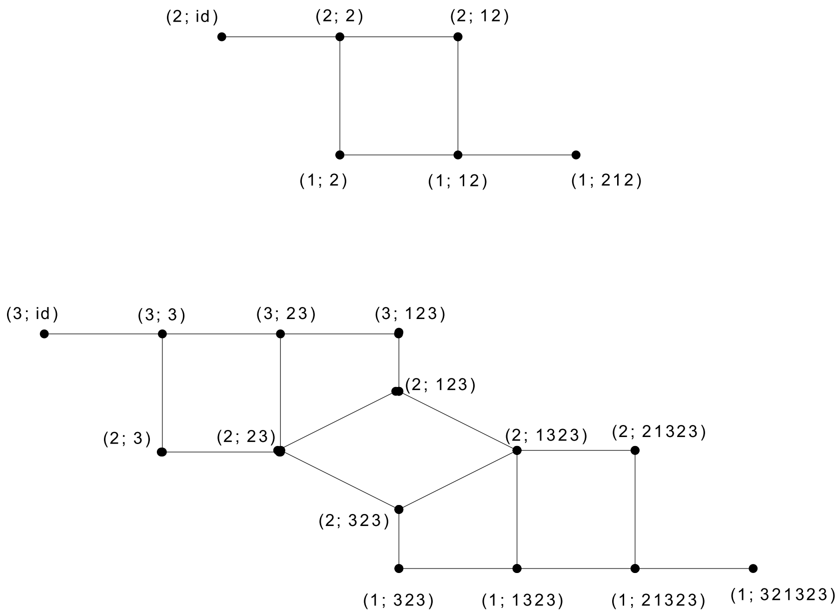


**Figure 8.** Hasse diagrams for the $su(3)$ and $su(4)$ fundamental posets. Next to the vertices, the corresponding integrity basis elements are indicated, by their shapes and a Weyl group element that maps their shape to their weight. For example, (2; 123) indicates shape $\Lambda^2$ and weight $r_1r_2r_3\Lambda^2 = -\Lambda^1 + \Lambda^3$, i.e., $L_2a_1^{-1}a_3$.

known, a vector in an arbitrary irreducible representation of $A_r$ can be constructed from the vectors of the fundamental representations, which are in turn constructible from the vectors of the first fundamental (basic) representation. The latter can be labelled by single boxes, numbered from 1 to $r + 1$. Totally antisymmetric $j$-fold tensor products of the basic vectors yield the vectors of the fundamental representation of highest weight $\Lambda^j$. Symmetrizing these, according to the rows of a fixed Young tableau, produces the vectors of the representation of highest weight equal to the tableau shape.

Consequently, the generating function X and the related fundamental poset $\mathcal{P}$ encode something of this construction[6]. Conversely, knowing how to construct the vectors from those of the fundamental representations, can tell us about the character generator X. As we will discuss below, the $G_2$ character generator was found this way in [2].

### *5.2. $B_2$*

The $A_r$ case is simple. All fundamental representations are minuscule, i.e., their weights form a single Weyl orbit $W\Lambda^j$. This means, in particular, that there are no inside generators in the case of $A_r$.

For the algebra $B_2$, however, there is one inside generator, $z := \bar{D}_2L_1a_1^{-1}a_2^2 = L_1$. From the expression (110), it is clear that this inside generator $z$ is not treated in the same way as the outer generators $A, \ldots, H$. While X is linear in $z$, it contains arbitrarily high powers of each of the outside generators.

[6] Arguably, the most important use of the character generator is to tell us about this method of building vectors of highest-weight representations.

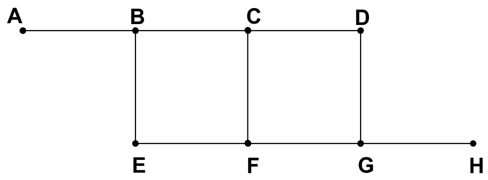


**Figure 9.** Hasse diagram of the fundamental-orbit poset of $B_2$.

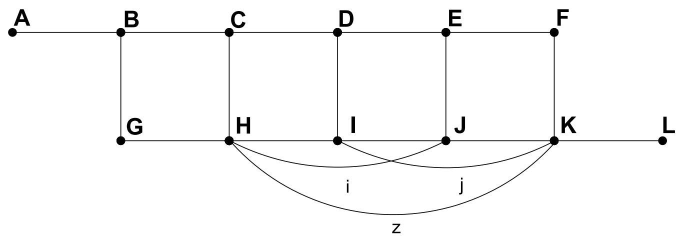


**Figure 10.** $G_2$ graph.

As in the $A_2$ case, however, the result (110) can be understood in terms of a graph related to a poset. We can use the same construction as for $A_2$, including the cover relations (130), as long as only the elements of the Weyl orbits $W\Lambda^j$ are included in the poset. For the $B_2$ case, the Hasse diagram of this poset is drawn in figure 9. We will call the poset so constructed the fundamental-orbit poset $\mathcal{P}_o$.

The connection of (110) to $\mathcal{P}_o$ is clear. If we modify the labelling of the maximal chains of $\mathcal{P}_o$ so that

$$\ell(\cdots > F > G > \cdots) = \cdots F(1+z)G\cdots, \tag{131}$$

then Baclawski's formula (127) still works. That is, in maximal chains, we introduce a labelling of the edge $\{F, G\}$ of the Hasse diagram connecting vertices $F$ and $G$, that corresponds to the cover relation $F > G$.

The extra labelling has a Demazure interpretation: $z = \bar{D}_2 F$ and

$$\widehat{D}_2\lfloor F\rfloor = \lfloor F\rfloor(1+\bar{D}_2F)\lfloor\hat{r}_iF\rfloor = \lfloor F\rfloor(1+z)\lfloor G\rfloor, \tag{132}$$

by equation (90).

The latter result shows that only the $\{F, G\}$ edge needs this extra factor, because no outer generator other than $F$ has a weight with a Dynkin label greater than 1. If generator $V$ has $i$th Dynkin label equal to 1, then $\bar{D}_iV = 0$. We can, therefore, extend the labelling to include all the edges of the Hasse diagram between vertices with weights in the same fundamental representation. Label with $(1+\bar{D}_jV)$ the edge $\{V, \hat{r}_jV\}$. For all cases considered so far, except $V = F$, this label is just 1.

### *5.3. $G_2$*

For $G_2$, $\|I_{\mathrm{in}}\| = 9$, so the situation becomes more complicated. That is made plain by looking at the final expression for X. The fundamental-orbit poset $\mathcal{P}_o$ has $\|I_{\mathrm{out}}\| = 12$ elements, and is easily constructed. If, as for $B_2$, we continue to label with $(1+\bar{D}_jV)$ the edge $\{V, \hat{r}_jV\}$, there are many terms recognizable in (114) as coming from the maximal chains of $\mathcal{P}_o$. However, there are many more terms in (114) than there are maximal chains in $\mathcal{P}_o$.

To proceed, we introduce new edges to the Hasse diagram of $\mathcal{P}_o$, and label the new edges as needed to produce the expression (114). The result is the graph of figure 10, where only the labels of the new edges are indicated. Three new edges are required: the $\{H, J\}$ edge with label $\hat{d}_1\bar{D}_2H = i$, $\{I, K\}$ with label $\hat{r}_2\hat{d}_1\bar{D}_2H = j$, and $\{H, K\}$ labelled by $\bar{D}_2\hat{d}_1\bar{D}_2H = z$. Consider now maximal paths (or walks) on this graph, from the beginning vertex $A$ to the end $F$. The terms of (114) can be put in one-to-one correspondence with the maximal paths, so that a formula of the Baclawski type can still be written, as long as the edge labels are included as factors.

The final element required is an explanation of the edge labels. They are sums of inside generators, but which ones? Their individual expressions in terms of Demazure operators are not particularly illuminating.

However, note that the new edges only relate outside generators to weights from the same fundamental representation, say $R(\Lambda)$, for some $\Lambda \in F$. Focus on a vertex $L^\Lambda a^{v\Lambda} = \hat{v}(L^\Lambda a^\Lambda)$, where $v \in W$. It is clear that any inside generators with weights in $\hat{d}_v a^\Lambda$ must appear as labels of edges ending on that vertex.

Consider the label $\hat{d}_1\bar{D}_2H = i$ of the new $\{H, J\}$ edge. Since $J = \hat{r}_1\hat{r}_2\hat{r}_1G$, we calculate

$$\hat{d}_{r_1r_2r_1}G = \hat{d}_1\hat{d}_2\hat{d}_1G =: \hat{d}_{1\,2\,1}G = i + z' + J. \tag{133}$$

The inside generator $i$ labels one edge ending on $J$, while $(1 + z')$ is the label of the other edge ending there.

This way, the labels of edges ending on a graph vertex can be found. To determine where the edges should begin, we simply reverse the process. Start with the outside generator of lowest weight, $L^\Lambda a^{w_L\Lambda} = \hat{w}_L(L^\Lambda a^\Lambda)$, $\Lambda \in F$. To work backwards, we need to consider Demazure operators like the $\hat{d}_m$, but where the role of the simple root $\alpha_m$ is taken by its negative $-\alpha_m$. We denote such a Demazure operator by $\hat{d}_{\underline{m}}$, and also use the convention that $\hat{d}_{\underline{\ell}\,\underline{m}} := \hat{d}_{\underline{\ell}}\hat{d}_{\underline{m}}$, etc.

For the $\{H, J\}$ edge, the generator of lowest weight is $L$. We calculate

$$\hat{d}_{\underline{2}\,\underline{1}\,\underline{2}\,\underline{1}}L = \hat{d}_{\underline{2}}\hat{d}_{\underline{1}}\hat{d}_{\underline{2}}\hat{d}_{\underline{1}}L = z + i + h + g + H. \tag{134}$$

Comparing (134) and (133), we see that only $i$ is common, and so $i$ will label the edge beginning at $H$ and ending on $J$.

This procedure works for all the new edges. It is easy to find

$$\begin{aligned} &\hat{d}_{2\,1\,2\,1}G = z + j + k + \ell + K, \\ &\hat{d}_{\underline{1}\,\underline{2}\,\underline{1}}L = j + z' + I, \hat{d}_{\underline{2}\,\underline{1}\,\underline{2}\,\underline{1}}L = z + i + h + g + H. \end{aligned} \tag{135}$$

The first two of these results confirm that the $\{I, K\}$ edge is labelled by $j$; the first and third give $z$ as the $\{H, K\}$ label.

The nontrivial $\bar{D}_iV$ part of the labels $(1 + \bar{D}_iV)$ for the edges $\{V, \hat{r}_iV\}$ of the Hasse diagram of $\mathcal{P}_o$ that is contained in our graph, can also be obtained this way.

Our graph, as shown in figure 10, has an obvious resemblance to that devised long ago by Baclawski and Towber [2], depicted in figure 11. Every element of the integrity basis (both inside and outside generators) is represented by a vertex in that graph. The Baclawski–Towber graph is not the Hasse diagram of a poset, however, but rather its generalization for a generalized poset. The generalization is necessary because an inner generator $\iota$ appears at most linearly in X. That means $\iota^2$ is an incompatible product. Incompatibilities correspond to incomparable elements of a poset, however, and the poset partial order $\geqslant$ obeys the reflexivity property: $x \geqslant x$, for all $x$ in a poset $\mathcal{P}$. For $x$ in a poset, then, $x$ is always comparable to $x$.

If the partial order $\geqslant$ is replaced by a binary relation $\gg$ without reflexivity; however, a generator $\iota$ can be incomparable to itself. All the inner generators $\iota$ do not obey $\iota \gg \iota$, and so $\iota^2$ can be an incompatible product. In the generalized poset, the inner generators are incompatible

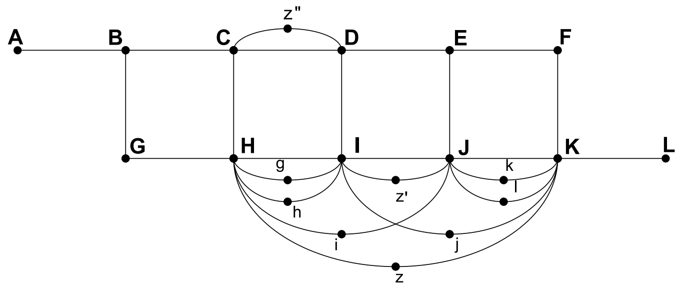


**Figure 11.** The $G_2$ generalized-poset graph of Baclawski and Towber [2].

with themselves, while the outer generators are not. As a consequence, the vertices of the corresponding graph are not all treated on an equal footing. With this modification, however, formulae like the poset ones written by Baclawski [1] can also be written for graphs of the type in figure 11.

In contrast, we prefer to work with a graph more closely related to the Hasse diagram of the fundamental-orbit poset $\mathcal{P}_o$. We do not increase the number of vertices by introducing new ones for every inner generator, i.e., for every integrity basis element with a weight not an element of a fundamental Weyl orbit $W\Lambda$, $\Lambda \in F$. Instead, we introduce edge labels involving the inner generators for the edges of the Hasse diagram of $\mathcal{P}_o$, and add new edges (only) with such labels. In our opinion, the resulting graph is simpler than that of [2]; compare figures 10 and 11. We will call our graph and its generalization to other simple Lie algebras the character-generator graph, and denote it by $\mathcal{G}_{\mathrm{X}}$.

We should point out, however, that just as Baclawski and Towber treat the vertices for inner and outer generators differently, we do not treat all the edges of $\mathcal{G}_{\mathrm{X}}$ on equal footing. An edge between two outer generators related by a primitive reflection $r_j$ gets special treatment. The 1 of the labels $(1 + \bar{D}_i V)$ of the edges $\{V, \hat{r}_i V\}$ must be added.

If we focus on the character generator of the algebra $G_2$ only, our result just amounts to a slight simplification of that of [2]. On the other hand, an important difference is revealed if we compare the methods used.

As was discussed above, semi-standard tableaux reveal the poset structure underlying the $su(r + 1) \cong A_r$ character generator. They also encode a construction of the vectors of an irreducible highest-weight representation, using as a basis the vectors of the fundamental representations. While writing down the vectors of a fixed representation is much more involved than finding its weights and multiplicities, doing the former does tell us about the latter, and so about the character of the representation. A general construction, for all highest weight representations can, therefore, reveal the structure of the character generator.

This construction of vectors is possible for any simple Lie algebra, providing a way to the character generator of that algebra. In [2], the authors defined what they called a shape algebra, which is useful for such constructions, but is framed in a more general context. More importantly for us, they constructed the required basis for $G_2$ explicitly, and were consequently able to draw the generalized poset graph of figure 11, and write the character generator X. Their work used the special relation of $G_2$ to the octonions $\mathbb{O}$, however[7]. It was therefore not able to yield results in a general form, useful for any simple Lie algebra. Their $G_2$ results

[7] The $G_2$ algebra is the algebra of derivations of $\mathbb{O}$. It is true that all the $A_r$, $B_r$, $C_r$, $D_r$, $E_6$, $E_7$, $E_8$, $F_4$ algebras can be related in a similar way to the four normed division algebras: $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ (see [3], e.g.). Even so, $G_2$ does not fit nicely into the pattern filled out by the others. For example, the $E$ and $F$ exceptional algebras are elements of the so-called magic square, while $G_2$ is not.

were not derived or written in terms of objects common to all simple Lie groups and/or their algebras, like the Weyl group, for example.

Our method, however, is essentially that of Gaskell [9], taking into account the poset structure of Baclawski [1]. As such it uses only general methods, involving Weyl groups and their Bruhat order, Demazure operators, and a total ordering of fundamental weights induced by a reduced decomposition of $w_L$. It therefore leads to results that we believe indicate the general form of the character generator for all simple Lie algebras.

### *5.4. General simple Lie algebras—possible universal picture*

Let us now sketch the construction of the character-generator graph $\mathcal{G}_{\mathrm{X}}$ in terms that apply to an arbitrary simple Lie algebra $X_r$.

We must emphasize that the construction outlined below has not been proven correct. All we can say at this point is that it works for the algebras we considered, and is expressed in universal terms. It therefore has a hope of applying to all simple Lie algebras.

First, construct the fundamental-orbit poset $\mathcal{P}_o$. Its elements are in one-to-one correspondence with the outside generators of the integrity basis (15) for X, and so with pairs $(\Lambda, \mu)$, where $\Lambda \in F$ and $\mu \in W_\Lambda \subset P_\Lambda$, the set of weights (of nonzero multiplicity) in the fundamental representation $R(\Lambda)$. We write

$$\mathcal{P}_o = \{[\Lambda, \mu; w] \mid \Lambda \in F, \mu \in P_\Lambda; w \in W \text{ such that } \mu = w\Lambda\}. \tag{136}$$

Note that only one Weyl group element is associated with each pair $(\Lambda, \mu)$, i.e. each element of $\mathcal{P}_o$. The choice of these Weyl elements is not unique; the different possible choices allow the same character generator X to be described by different posets.

To make one such choice, fix a reduced decomposition of $w_L$, and write it as

$$w_L := s_L s_{L-1} \cdots s_1, \tag{137}$$

where each $s_a$ is a primitive reflection of $W$, so that $L = \ell(w_L)$, the length of $w_L$. More generally, we will use

$$w_{L,a} := s_a s_{a-1} \cdots s_1. \tag{138}$$

Set the highest-weight elements of $\mathcal{P}_o$ to be $\left[\Lambda^j, \Lambda^j; w_{\max}^{(j)}\right]$, where $w_{\max}^{(j)}$ is the longest of the Weyl group elements $w_{L,a}$ fixing $\Lambda^j$: $w_{\max}^{(j)}\Lambda^j = \Lambda^j$. Then the remaining elements can be assigned Weyl group elements using the Bruhat order: $[\Lambda, \mu; w] > [\Lambda, \nu; v]$ if $w \prec v$.

The reduced decomposition of $w_L$ selected also induces a total order $\geqslant$ on the fundamental weights of $F$. Let $\Rightarrow$ denote its cover relations. We put $\Lambda^j > \Lambda^i$ if $\ell\left(w_{\max}^{(j)}\right) < \ell\left(w_{\max}^{(i)}\right)$.

The partial order of $\mathcal{P}_o$ can then finally be fully defined by the cover relations:

$$\begin{aligned} [\Lambda, \mu; w] \to [\Lambda', \nu; v], \quad & \text{if } \Lambda = \Lambda' \text{ and } w \leftarrow v, \\ & \text{or if } \Lambda \Rightarrow \Lambda' \text{ and } w = v. \end{aligned} \tag{139}$$

Let $\mathrm{E}(\mathcal{G})$ and $\mathrm{V}(\mathcal{G})$ indicate the edge set and the vertex set, respectively, of a graph $\mathcal{G}$. The character-generator graph $\mathcal{G}_{\mathrm{X}}$ is built on the skeleton $\mathcal{H}(\mathcal{P}_o)$. More precisely,

$$V(\mathcal{G}_{\mathrm{X}}) = V(\mathcal{H}(\mathcal{P}_o)), \qquad E(\mathcal{G}_{\mathrm{X}}) \supset E(\mathcal{H}(\mathcal{P}_o)). \tag{140}$$

Label the vertices of the Hasse diagram $\mathcal{H}(\mathcal{P}_o)$ of $\mathcal{P}_o$ using

$$\ell([\Lambda, \mu; w]) = L^\Lambda a^\mu. \tag{141}$$

We will also label the edges of the resulting character-generator graph, using Demazure objects. First, all edges of the $\mathcal{P}_o$ Hasse diagram are labelled by 1. Additional labels are introduced as follows, and they will add to the 1s already present, or label new edges of $\mathcal{G}_X \supset \mathcal{H}(\mathcal{P}_o)$, when they do not vanish.

The edge labels and the 'new' edges, the elements of $\mathrm{E}(\mathcal{G}_\mathrm{X})\backslash\mathrm{E}(\mathcal{H}(\mathcal{P}_o))$, are found using Demazure calculations. Suppose that $T = L^\Lambda$ and $B = \hat{w}_L T$ indicate the top and bottom vertices of the same shape $\Lambda \in F$. Consider the vertices $V_1$ and $V_2$, with $V_1 > V_2$ in $\mathcal{H}(\mathcal{P}_o)$. Suppose further that $V_1 = \hat{b}_1 B$, and $V_2 = \hat{t}_2 T$, with $b_1, t_2 \in W$. Calculate $\hat{d}_{t_2} T$ and $\hat{d}_{\underline{b_1}} B$. Denote the sum of terms common to both as

$$d(V_1, V_2) := \hat{d}_{\underline{b_1}} B \cap \hat{d}_{t_2} T. \tag{142}$$

If $d(V_1, V_2) \neq 0$, then $\{V_1, V_2\}$ will belong to $\mathrm{E}(\mathcal{G}_\mathrm{X})$.

The labels of the edges of $\mathcal{G}_\mathrm{X}$ are given by

$$\tilde{\ell}(\{V_1, V_2\}) = \begin{cases} 1 + d(V_1, V_2), & \text{if} \quad \{V_1, V_2\} \in \mathrm{E}(\mathcal{H}(\mathcal{P}_o)); \\ d(V_1, V_2), & \text{if} \quad \{V_1, V_2\} \notin \mathrm{E}(\mathcal{H}(\mathcal{P}_o)). \end{cases} \tag{143}$$

We can write a formula analogous to (127) for the general case if we consider $\mathcal{G}_\mathrm{X}$ the Hasse diagram of a poset $\mathcal{P}_\mathrm{X}$. The new poset $\mathcal{P}_\mathrm{X}$ has the same elements as the fundamental-orbit poset $\mathcal{P}_o$, but its cover relations are those of $\mathcal{P}_o$ augmented by those encoded in the new edges of $\mathcal{G}_\mathrm{X}$.

Maximal chains in $\mathcal{P}_\mathrm{X}$ are relevant here, but their labels must include the edge labels as factors, along with those of the vertices. We define

$$\tilde{\ell}(\cdots V_1 > V_2 \cdots) := \cdots \tilde{\ell}(V_1)\tilde{\ell}(\{V_1, V_2\})\tilde{\ell}(V_2) \cdots . \tag{144}$$

The symbol $\tilde{\ell}$ indicates the labelling of $\mathcal{G}_\mathrm{X}$, to distinguish it from the labelling $\ell$ of $\mathcal{P}_o$.

We need the extended poset $\hat{\mathcal{P}}_\mathrm{X}$, and its labelling. But its labelling is trivially different from that of $\mathcal{P}_\mathrm{X}$: vertices $\hat{1}$ and $\hat{0}$, and the two extra edges involving them, are all assigned 1 as labels. We will also use $\tilde{\ell}$ for the labels of $\hat{\mathcal{P}}_\mathrm{X}$.

Incompatible products are treated in (127) using linkings of the extended poset $\hat{\mathcal{P}}$ and the resulting descent sets, defined in (125). In the general case, only incompatibilities between two outside generators need to be handled this way. Therefore, it is the linking of $\hat{\mathcal{P}}_o$ that is relevant. Suppose $m$ is a maximal chain in $\hat{\mathcal{P}}_\mathrm{X}$. Then we define

$$\mathcal{DS}_o(m) := \{x_i \mid 0 < i < n \text{ and } (x_{i-1} > x_i > x_{i+1}) \in \mathrm{Link}^-(\hat{\mathcal{P}}_o)\}. \tag{145}$$

Finally, we are able to write

$$\mathrm{X} = \sum_{m \in \mathrm{Max}(\hat{\mathcal{P}}_\mathrm{X})} \lfloor \tilde{\ell}(m) \rfloor \ell(\mathcal{DS}_o(m)). \tag{146}$$

Here the shorthand notation of (81) and (109) only applies to the vertex factors:

$$\lfloor \tilde{\ell}(\cdots V_1 > V_2 \cdots) \rfloor := \cdots \lfloor \tilde{\ell}(V_1) \rfloor \tilde{\ell}(\{V_1, V_2\}) \lfloor \tilde{\ell}(V_2) \rfloor \cdots . \tag{147}$$

The formula (146) for the character generator X is one of our main results. Hopefully, our conjecture generalizes Baclawski's formula (127) so that it can be applied to any simple Lie algebra.

## 6. Conclusion

Let us first summarize our main results. A new, universal formula was derived for the character generator of a simple Lie algebra. The character generator X is expressed as the ratio $\mathrm{X} = \mathrm{Y}/\mathrm{Z}$, with the simple denominator given by (12), and the numerator by (22), or, equivalently, by (27) and (25). The new formula does not involve a sum over the Weyl group, and so is a simplification of the Patera–Sharp formula. It a also makes clear the distinct roles of the inside and outside generators, and can serve as a guide to incompatible products, as section 3 indicates.

In the second part of this paper, the general, Demazure methods of Gaskell [11] were exploited, and connected with the (generalized-)posets underlying the character generator [1, 2, 15–17, 23]. Graphs were found that are simplified versions of those introduced by Baclawski–Towber [2], from which the character generators can be determined easily. In particular, the required labelling of the edges of these graphs is given by simple Demazure calculations. By combining the universal Demazure–Gaskell approach with the graph structure of the character generators, we were able to formulate a general conjecture, that we hope is applicable to all simple Lie algebras. Thus, non-negativity did not have to be sacrificed to attain universality. The general formula is equation (146), and it is explained in the rest of subsection 5.4.

Our second main result is only a conjecture, and clearly lacks rigour. Possible future work therefore includes proving (146). Induction may be helpful, and one might be able to extend our result to a generalization of the character generator:

$$\mathrm{X}_w := \widehat{D}_w \mathrm{H}, \qquad w \in W. \tag{148}$$

Here $\mathrm{X}_{w_L} = \mathrm{X}$.

Alternatively, the general character formulae of Littelmann, written in terms of minimal defining chains [18] and Lakshmibai–Seshadri paths [19], could provide another route to the character generators, and a proof.

The character-generator formulae could also be investigated to see what they tell us directly about the characters themselves. Can a new character formula be written? Can one derive new identities involving the characters? The relation of the character generators and the corresponding integrity bases to bases of states (or vectors) in irreducible highest-weight representations should also be understood.

There is a fundamental correspondence found by C Greene (see [4] for a review) that associates with every finite poset a Young tableau, or Ferrers shape. It would be interesting to try to apply the correspondence, or a modified version thereof, to the (generalized-)posets underlying the character generators. A significantly more economical presentation of the character generators might result. We suspect that in the simplest cases, the early results of Stanley [23] and King [15] would be recovered.

Let us conclude by describing an application of character generators that was the original motivation for this work. Two-dimensional conformal field theories [7] have been intensely investigated for quite some time now. Important examples, the Wess–Zumino–Witten models, are intimately related to simple Lie algebras. Their so-called modular data (see [7, 9, 10]), including their fusion eigenvalues, are fundamental characteristics. But the fusion eigenvalues of Wess–Zumino–Witten models coincide with the characters of simple Lie algebras, evaluated at certain finite-order elements of the corresponding Lie group. Thus the character generator of a simple Lie algebra can be used to study Wess–Zumino–Witten fusion eigenvalues. One of us (MW) hopes to make progress in this direction. The work [20] studied character generators for elements of finite order and so should be helpful.

## Acknowledgments

This research was supported by NSERC of Canada. We thank the Department of Applied Mathematics at the University of Western Ontario for its hospitality. For their comments, we are grateful to Chris Cummins, Pierre Mathieu, and especially to Terry Gannon and Dave Morris. MW also thanks the Perimeter Institute, where some of this work was done.

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